Analysis and Design of Cognitive Radio Networks and Distributed Radio Resource Management Algorithms

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(Abstract)

Cognitive radio is frequently touted as a platform for implementing dynamic distributed radio resource management algorithms. In the envisioned scenarios, radios react to measurements of the network state and change their operation according to some goal driven algorithm. Ideally this flexibility and reactivity yields tremendous gains in performance. However, when the adaptations of the radios also change the network state, an interactive decision process is spawned and once desirable algorithms can lead to catastrophic failures when deployed in a network.

This document presents techniques for modeling and analyzing the interactions of cognitive radio for the purpose of improving the design of cognitive radio and distributed radio resource management algorithms with particular interest towards characterizing the algorithms' steady-state, convergence, and stability properties. This is accomplished by combining traditional engineering and nonlinear programming analysis techniques with techniques from game to create a powerful model based approach that permits rapid characterization of a cognitive radio algorithm's properties. Insights gleaned from these models are used to establish novel design guidelines for cognitive radio design and powerful low-complexity cognitive radio algorithms.

This research led to the creation of a new model of cognitive radio network behavior, an extensive number of new results related to the convergence, stability, and identification of potential and supermodular games, numerous design guidelines, and several novel algorithms related to power control, dynamic frequency selection, interference avoidance, and network formation. It is believed that by applying the analysis techniques and the design guidelines presented in this document, any wireless engineer will be able to quickly develop cognitive radio and distributed radio resource management algorithms that will significantly improve spectral efficiency and network and device performance while removing the need for significant post-deployment site management.

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Throughout this time my wife, Preethi, my father, Dr. James B. Neel, and my brother, David, have been a source of love, support, and encouragement.

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Chapter 1: Cognitive Radio

"*Cogito, ergo sum.* [I think, therefore I am]" – René Descartes

Aided by advances in processors, RF technology, and software, software radio technology has rapidly progressed since the coining of the term "software radio" by Joe Mitola in 1991 [Mitola_00]. Software radio (SDR) currently forms the core of the US military's multi-billion-dollar-a-year Joint Tactical Radio System (JTRS) which has resulted in SDRs being fielded by General Dynamics (DMR [GDDS]), Thales (JEM [Thales]), and Harris (RF-300M-HH [Harris]), to name a few. Beyond the military, commercial standards are beginning to be preferably implemented in software (802.16 [Picochip]) and commercial base stations are being implemented as software radios (Vanu [Vanu]). The reality of software radio and the support for moving a single radio through multiple standards has led the Institute of Electrical and Electronics Engineers (IEEE) to begin standardizing vertical handoffs between networks employing different standards (802.21 [802.21]).

However, the numerous envisioned applications for SDR – multiband multimode radio, porting waveforms across platforms, over-the-air updates – are accompanied by the numerous envisioned problems – viruses or worms that render the radio unusable, unforeseen software/hardware combinations that turn radios into jammers, and cell phones that crash to reveal a "blue screen of death." Accordingly, SDR research and development has focused as much on overcoming the problems created by SDR as it has the opportunities and realization of SDR.

Now consider a radio that autonomously detects and exploits empty spectrum to increase your file transfer rate. Suppose this same radio could remember the locations where your calls tend to drop and arrange for your call to be serviced by a different carrier for those locations. These are some of the ideas motivating the development of *cognitive radio*[1]. In effect, a cognitive radio is a software radio whose control processes leverage situational knowledge and intelligent processing to work towards achieving some goal related to the

[1] This term, too, was coined by Mitola in 1999 [Mitola_99].

needs of the user, application, and/or network. Arising from a logical evolution of the control processes of a software radio, cognitive radio presents the possibility of numerous revolutionary applications.

Opportunistic spectrum utilization can find available spectrum in a crowded network, leading to 10-fold gains in capacity [Marshall_05a]. By learning their environment, cognitive radios can dramatically improve link reliability and help networks autonomously improve coverage and capacity. True radio interoperability can be achieved when radios learn to autonomously negotiate services and protocols. Smart collaborative signaling techniques promise significant range extension and data-rate increases. Advanced network topologies can dramatically extend coverage and increase bandwidth. The global roaming of radios can be dramatically simplified when a radio is responsible for autonomously detecting the location specific operating requirements. Autonomous determination of bandwidth requirements and spectrum availability will greatly enhance the opportunities for rapid reallocation of spectrum resources. Finally, smart spectrum use can overcome the deficiencies of inexpensive analog components allowing lower-priced radios to be fielded [Marshall_05a].

But what's to say that cognitive radios will not act maliciously – opportunistic spectrum use into spectrum bullying? Given the infinite number of environments that a radio will encounter and a design that how can we hope to verify that the radio will behave as intended? How can we be certain that radios will be able to even recognize the opportunities they are presented? What if the interaction of several seemingly benign algorithms yield disastrous network behavior – something that seems all too possible once selfish radios are competing for spectrum?

This work concentrates on this last problem – the interaction of cognitive radios in distributed radio resource management settings – by developing techniques for modeling and analyzing cognitive radio algorithms to determine steady-states, convergence, and stability and by developing frameworks for designing cognitive radio algorithms that yield good performance for the radio and for the network. Beyond cognitive radio, the techniques developed and presented in this work can also be extended to the modeling, analysis, and design of distributed and automated radio resource management. Serving as a foundation on which to build the subsequent models, analysis techniques and development frameworks, this chapter focuses on the concept, implementation, and applications of cognitive radio and is organized as follows. Section 1.1 formally defines cognitive radio and differentiates cognitive radio from some closely related terms. Section 1.2 discusses high-level implementation aspects of cognitive critical to understanding the analysis of interactive cognitive radio. Section 1.3 discusses some of the frequently discussed applications of cognitive radio and some limited current deployments of cognitive radio. Section 1.4 presents some of the major technical hurdles that must be cleared for widespread deployment of cognitive radio. Section 1.5 briefly overviews the objectives and original contributions of this work. Section 1.6 presents work related to the big-picture objectives of this work and outlines the material to be presented over the remainder of the text.

1.1 Basic Cognitive Radio Concepts

While the cognitive radio community has had significant success popularizing the concept of cognitive radio and developing prototypes, applications, and critical components, the community has had a surprisingly difficult time agreeing upon exactly what is and is not a cognitive radio beyond. Perhaps echoing the sentiment of former Supreme Court Justice Potter Stewart, many members of the cognitive radio community believe that "they know it when they see it," even if a precise definition is ineffable. Some have succeeded in formulating a definition of cognitive radio but have found their definitions at significant variance with others' definitions.

While these definitions are likely to converge over time from either an international consensus (the goal of IEEE 1900.1 group) or from a *de facto* definition taken from the first cognitive radio to dominate the market, this dissertation must plunge ahead with some formalization of cognitive radio and related concepts to formally analyze their interactions. To that end, the remainder of this introductory section presents a definition of cognitive radio that is hopefully suitably encompassing and discriminating for other researchers to use and reasonably well-justified by its preceding discussion. To enhance the offered definition's usefulness, terms frequently discussed in relation to cognitive radio are subsequently defined and differentiated from cognitive radio.

1.1.1 Defining "Cognitive Radio"

Tautologically, a cognitive radio could be defined as "A radio that is cognitive," or paraphrasing Descartes, "*Cogitat, ergo est* cognitive radio."[2] In the absence of a Turing test for radios, applying this definition is nontrivial and implies a level of functionality that many researchers consider excessive. Indeed, while many researchers and public officials agree that upgrading a software radio's control processes will add significant value to software radio, there is currently some disagreement over how much "cognition" is needed which results in disagreement over the precise definition of a cognitive radio. The following provides some of the more prominently offered definitions of cognitive radio.

In the 1999 paper that first coined the term "cognitive radio", Joseph Mitola III defines a cognitive radio as [Mitola_99]: "*A radio that employs model based reasoning to achieve a specified level of competence in radio-related domains*."

However, in his recent popularly cited paper that surveyed the state of cognitive radio, Simon Haykin defines a cognitive radio as [Haykin_05]: "*An intelligent wireless communication system that is aware of its surrounding environment (i.e., outside world), and uses the methodology of understanding-by-building to learn from the environment and adapt its internal states to statistical variations in the incoming RF stimuli by making corresponding changes in certain operating parameters (e.g., transmit-power, carrier-frequency, and modulation strategy) in real-time, with two primary objectives in mind:*

- *highly reliable communications whenever and wherever needed;*
- *efficient utilization of the radio spectrum.*

[2] It thinks, therefore it's a cognitive radio.

Coming from a background where regulations focus on the operation of transmitters, the FCC has defined a cognitive radio as [FCC_05]: "*A radio that can change its transmitter parameters based on interaction with the environment in which it operates.*"

Meanwhile, the other primary spectrum regulatory body in the US, the NTIA [NTIA_05], adopted the following definition of cognitive radio that focuses on some of the applications of cognitive radio: "*A radio or system that senses its operational electromagnetic environment and can dynamically and autonomously adjust its radio operating parameters to modify system operation, such as maximize throughput, mitigate interference, facilitate interoperability, and access secondary markets.*"

The international spectrum regulatory community in the context of the ITU Wp8A working document is currently working towards a definition of cognitive radio that focuses on capabilities as follows: "*A radio or system that senses and is aware of its operational environment and can dynamically and autonomously adjust its radio operating parameters accordingly.*"

While aiding the FCC in its efforts to define cognitive radio, IEEE USA offered the following definition [IEEEUSA_03]: "*A radio frequency transmitter/receiver that is designed to intelligently detect whether a particular segment of the radio spectrum is currently in use, and to jump into (and out of, as necessary) the temporarily-unused spectrum very rapidly, without interfering with the transmissions of other authorized users.*"

The broader IEEE tasked the IEEE 1900.1 group to define cognitive radio which has the following working definition [IEEE 1900.1]: "*A type of radio that can sense and autonomously reason about its environment and adapt accordingly. This radio could employ knowledge representation, automated reasoning and machine learning mechanisms in establishing, conducting, or terminating communication or networking functions with other radios. Cognitive radios can be trained to dynamically and autonomously adjust its operating parameters.*"

Likewise, the SDR Forum participated in the FCC's efforts to define cognitive radio and has established two groups focused on cognitive radio. The Cognitive Radio Working Group focused on identifying enabling technologies uses the following definition: *"A radio that has, in some sense, (1) awareness of changes in its environment and (2) in response to these changes adapts its operating characteristics in some way to improve its performance or to minimize a loss in performance."*

However, the SDR Forum Special Interest Group for Cognitive Radio, which is developing cognitive radio applications, uses the following definition: *"An adaptive, multi-dimensionally aware, autonomous radio (system) that learns from its experiences to reason, plan, and decide future actions to meet user needs."*

Finally, the author of this text participates in the Virginia Tech Cognitive Radio Working Group which has adopted the following capability-focused definition of cognitive radio [VT CRWG]: *"An adaptive radio that is capable of the following:*

a) awareness of its environment and its own capabilities,

b) goal driven autonomous operation,

c) understanding or learning how its actions impact its goal,

d) recalling and correlating past actions, environments, and performance."

While it appears to be unlikely that there will be a harmonization of these definitions in the near future, an examination of the salient functionalities of these definitions, as summarized in Table 1.1, reveals some commonalities among these definitions. First, all of these definitions assume that cognition will be implemented as a control process, presumably as part of a software defined radio. Second, all of the definitions at least imply some capability of autonomous operation. Finally, the following are some general capabilities found in all of the definitions:

1. **Observation –** whether directly or indirectly, the radio is capable of acquiring information about its operating environment.
2. **Adaptibility –** the radio is capable of changing its waveform.

3. **Intelligence** – the radio is capable of applying information towards a purposeful goal.

Table 1.1: Cognitive Radio Definition Matrix.

Definer	Adapts (Intelligently)	Autonomous	Can sense Environment	Transmitter	Receiver	"Aware" Environment	Goal Driven	Learn the Environment	"Aware" Capabilities	Negotiate Waveforms	No interference
FCC	•	•	•	•							
Haykin	•	•	•	•	•	•	•	•			
IEEE 1900.1	•	•	•	•	•						
IEEE USA	•	•	•	•	•	•					•
ITU-R	•	•	•	•	•	•					
Mitola	•	•	•	•	•	•	•	•	•	•	
NTIA	•	•	•	•	•	•	•				
SDRF CRWG	•	•	•	•	•		•				
SDRF SIG	•	•	•	•	•	•	•	•	•		
VT CRWG	•	•	•	•	•	•	•	•	•		

Note that this definition of intelligence[3] implies that even those definitions that do not explicitly mention a goal (or provide a specific goal such as performance) still implicitly require the existence of some goal for intelligent adaptation. By using only these common features of all these definition we arrive at the definition of cognitive radio given in Definition 1.1.

Definition 1.1: *Cognitive Radio* (*)[4]
A *cognitive radio* is a radio whose control processes permit the radio to leverage situational knowledge and intelligent processing to autonomously adapt towards some goal.

[3] Intelligence as defined by [American Heritage_00] as *"The capacity to acquire and apply knowledge, especially toward a **purposeful goal**."* The definition for intelligence as applied to cognitive radio differs only in that the acquisition of knowledge has been subsumed into the observation process.
[4] The asterik denotes that this definition is original to the author. Throughout this document when chapters present both original and prior work by others, original definitions and theorems are noted by an asterik.

Throughout the remainder of this text Definition 1.1 will be what is meant when the phrase "cognitive radio" is used. When different capabilities (sometimes more, sometimes less, sometimes more specific) are required, we will make use of different terms defined in the following section.

1.1.2 Related Terms

As part of our discussion of cognitive radio, we will find it useful to rigorously define some related terms. Specifically we will find it useful to make use of the terms software defined radios, policy based radios, procedural radios, and ontological radios.

The logical authority for a definition of software defined radio, the Software Defined Radio Forum defines a *software defined radio* (SDR) as shown in Definition 1.2 [SDR Forum_05].

Definition 1.2: *Software Defined Radios* (*SDRs*)
"Radios that provide **software control** of a variety of modulation techniques, wide-band or narrow-band operation, communications security functions (such as hopping), and waveform requirements of current and evolving standards over a broad frequency range. The frequency bands covered may still be constrained at the front-end, however, requiring a switch in the antenna system."

While others[5] consider Definition 1.2 to be that of a *"software controlled radio,"* this text will utilize the definition of SDR provided by the SDR Forum and will refer to radios whose functionality is primarily realized in software as "*software implemented radios*." While a radio could be implemented via software or hardware and controlled via software, the emphasis on software control is important to the cognitive radio concept as software control permits rapid adaptation of the radio's operation – perhaps as short as the time required to readdress a program address counter – and provides a logical mechanism on which to implement a cognitive radio's "*control processes [that] permit the radio to leverage situational knowledge and intelligent processing to autonomously adapt towards achieving some goal*." It should be pointed out that while many treat that cognitive radio as an SDR with enhanced control processes (as was done in the introduction to this chapter), some researchers correctly emphasize that a cognitive radio

[5] Here, the term "others" includes this author.

need not be implemented on an SDR as the control processes could be implemented in hardware and indeed have implemented a hardware cognitive radio [Rondeau_04]. However, such "hardware controlled cognitive radios" appear likely to share the fate of Babbage's Analytical Engine due to the far greater flexibility and ease of programming provided by SDR cognitive radios.

Although the definition of the term *waveform* is a frequent point of discussion at conferences due to variances in usage, we have used the term repeatedly throughout the preceding and hope its usage has been clear from context up to this point. However, as we just wrote "waveform requirements" as part of a formal definition, completeness dictates we formally define waveform.

Definition 1.3: *Waveform* (*)
A protocol that specifies the shape of an electromagnetic signal intended for transmission by a radio.

Implicit to Definition 1.3 is the fact that a waveform is not solely defined by its physical layer algorithms. If specified as part of the protocol, then link layer, network layer, transport layer, and application layer algorithms will all influence the shape of the electromagnetic signal. However, not all waveforms specify algorithms at all layers. For example the FM broadcast radio waveform is a purely physical layer standard. In general, the only influence of signal shape that is excluded from the term "waveform" is the information bits being carried by the signal.

Because cognitive radios and SDRs could conceivably be configured (or autonomously configure themselves) to implement almost any waveform, spectrum regulators need some mechanism to ensure that cognitive and software defined radios have a limited impact on licensed systems. To provide this mechanism, many researchers have proposed the use of *policy radios*. The IEEE 1900.1 group currently defines a *policy radio* as given in Definition 1.4 [IEEE 1900.1].

Definition 1.4: *Policy Radio*
"A radio that is governed by a set of rules for choosing between different waveforms. The definition and implementation of these rules can be:

- during the manufacturing process
- during configuration of a device by the user;

- during over-the-air provisioning; and/or
- by over-the-air control."

Particularly for cognitive radios, it is convenient to refer to the set of set of rules as the radio's *policy* and a policy radio that is also a cognitive radio as a *cognitive policy radio*. In practice, a policy might specify a spectral mask which defines a set of maximum transmission powers for a number of different frequency bands that are specific to a particular location. Then as the policy-based radio is moved around the world, the policy-based cognitive radio would be responsible for inferring and applying the policy that applies to its particular location, perhaps via GPS, a radio environment map [Zhao_06], or from a primary spectrum holder. Because of the needs of spectrum regulators and primary spectrum holders, it is expected that all cognitive radios will eventually be cognitive policy radios. In fact, the incorporation of policy into cognitive radio has been a major focus of DARPA's xG program [Marshall_06].

Especially assuming the use of an SDR platform, the actual implementation of the cognitive and policy-related processes permits significant variation. The more traditional approach implements the control processes in a procedural language, such as C, where the adaptations spawned from specific observations can be traced to a specific pre-coded function. Such a cognitive radio is termed a *procedural cognitive radio* which is more formally defined as given in Definition 1.5. [Neel_06]

Definition 1.5: *Procedural Cognitive Radio* (*)
A cognitive radio whose adaptations are determined by hard coded algorithms and informed by observations.

Most implemented cognitive radio prototypes exhibit a significant degree of hard coding in their adaptation algorithms and are thus procedural cognitive radios. This includes Adapt4's xG1 cognitive radio [Adapt4_06] which implements a dynamic frequency selection algorithm to adapt around legacy systems and CWT's cognitive radio [Rondeau_04] which utilizes a genetic algorithm [6] to generate adaptations from

[6] A genetic algorithm is a search algorithm from optimization theory which generates sequences of candidate solutions by using an algorithm based on the gene theory of evolution. As such the algorithm exhibits both randomness (e.g., "mutations" and random "chromosome" cross-overs to generate "children")

observations. Due to its significant parameterization, the CWT radio is significantly more flexible than the Adapt4 radio and is significantly less hard-coded, but it remains a procedural cognitive radio. Also the CWT radio illustrates that though procedural, the adaptations of a procedural radio may be nondeterministic. Thus when possible we will distinguish between deterministic and nondeterministic procedural cognitive radios.

However, as discussed in the text related to Fig 6 in the April 1900.1 draft, many researchers do not believe that a radio whose adaptations are determined by hard coded algorithms constitutes a cognitive radio. This is primarily because these researchers utilize a definition of cognitive radio which emphasizes a different implementation approach that utilizes a form of artificial intelligence. To provide this intelligence in a radio, [Mitola_00] proposes model-based reasoning using the Radio Knowledge Representation Language (RKRL) and [Baclawski_05] and [Kokar_06] propose ontological reasoning for cognitive radio applications. As defined by the IEEE 1900.1, an *ontology* is "the representation of terms in a vocabulary and their inter-relationships." As such, RKRL would satisfy the IEEE 1900.1 definition of an ontology, thus these two different approaches could be said to be of the same "genus" (a cognitive radio that employs ontologies) if not the same "species" of ontology. The primary difference between the two approaches being the former's usage of a vocabulary restricted to radio information while the latter uses the Web Ontology Language (OWL) to extend the radio's knowledge base beyond radio-specific information.

In the context of cognitive radio, ontologies are intended to permit a reasoning engine to make inferences about the radio's operating environment and what actions would be in the cognitive radio's interest. Such an ontological approach has been employed by the DARPA xG program to demonstrate the feasibility of multiple cognitive radios implementing dynamic frequency selection (DFS). For the purposes of this text, we consider cognitive radios that adapt based on the decisions of a reasoning engine and

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and determinism (e.g., picking "surviving populations" based on their "fitness" which for cognitive radios is expressed in terms of the goal of the cognitive radio). More information on genetic algorithms and CWT's genetic algorithm cognitive radio is available in [Reiser04].

incorporate ontologies to be *ontological cognitive radios* which we formally define in Definition 1.6. [Neel_06]

Definition 1.6: *Ontological Cognitive Radio* (*) A cognitive radio whose adaptations are determined by some reasoning engine which is guided by its ontological knowledge base (which is informed by observations).

Though this distinction is blurred for nondeterministic procedural cognitive radios, e.g., the biologically inspired cognitive radio [Rondeau_04], an ontological cognitive radio could conceptually perform both much better and much worse than a procedural cognitive radio. Whereas for a procedural cognitive radio we typically know what action the radio will take when a known collection of observations are input to the radio, the same can not be said for an ontological cognitive radio as it truly has a mind of its own (the reasoning engine). Instead, for an ontological cognitive radio we only know that the radio will take an action the radio believes (or the engine calculates) furthers the radio's goal. While this imprecision appears to be a significant hurdle to analyzing the interactions of cognitive radios, we introduce techniques for analyzing these radios beginning in Chapter 4.

1.2 Cognitive Radio Implementation and Standardization

The differences in the definitions for cognitive radio can be largely attributed to differences in the expectations of the functionality that a cognitive radio will exhibit. In his dissertation [Mitola_00], Joseph Mitola III considers the nine levels of increasing cognitive radio functionality shown in Table 1.1, ranging from a software radio to a complex self-aware radio.

Table 1.1: Levels of cognitive radio functionality. Adapted from Table 4-1 [Mitola_00].

Level	Capability	Comments
0	Pre-programmed	A software radio
1	Goal Driven	Chooses Waveform According to Goal. Requires Environment Awareness.
2	Context Awareness	Knowledge of What the User is Trying to Do
3	Radio Aware	Knowledge of Radio and Network Components, Environment Models
4	Capable of Planning	Analyze Situation (Level 2& 3) to Determine Goals (QoS, power), Follows Prescribed Plans
5	Conducts Negotiations	Settle on a Plan with Another Radio
6	Learns Environment	Autonomously Determines Structure of Environment
7	Adapts Plans	Generates New Goals
8	Adapts Protocols	Proposes and Negotiates New Protocols

As a reference for how a cognitive radio could achieve these levels of functionality, [Mitola_00] introduces the cognition cycle, shown in Figure 1.1, as a "top-level control loop for cognitive radio." In the cognition cycle, a radio receives information about its operating environment (**Outside world**) through direct observation or through signaling. This information is then evaluated (**Orient**) to determine its importance. Based on this valuation, the radio determines its alternatives (**Plan**) and chooses an alternative (**Decide**) in a way that presumably would improve the valuation. Assuming a waveform change was deemed necessary, the radio then implements the alternative (**Act**) by adjusting its resources and performing the appropriate signaling. These changes are then reflected in the interference profile presented by the cognitive radio in the **Outside world**. As part of this process, the radio uses these observations and decisions to improve the operation of the radio (**Learn**), perhaps by creating new modeling states, generating new alternatives, or creating new valuations.

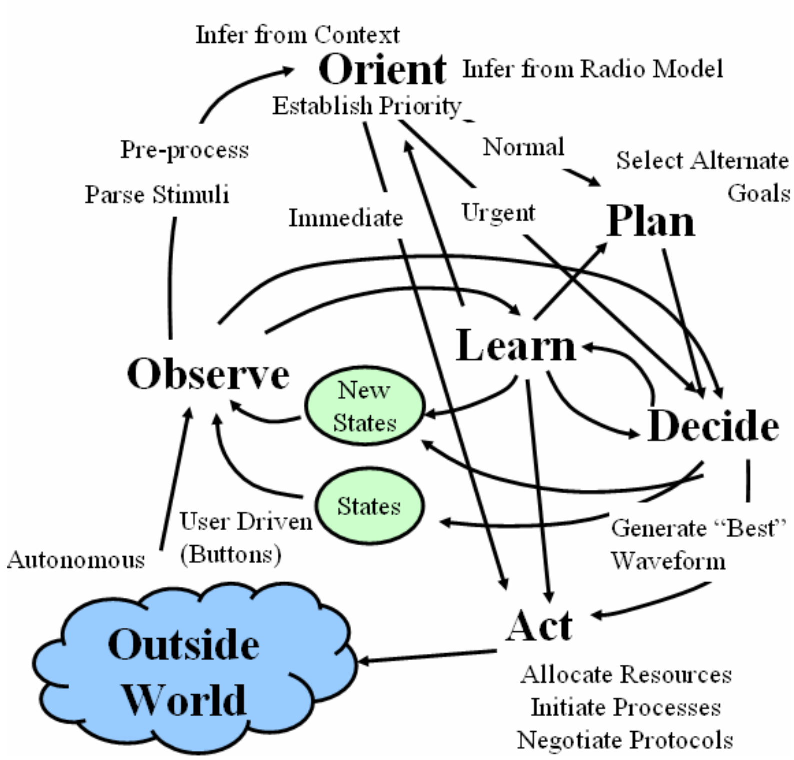

Figure 1.1: Cognition cycle. Reproduced from Figure 4-2 [Mitola_00]

As the learning process can be quite cycle intensive and is not necessary for many of the envisioned applications and as artificial intelligence is not yet ripe for deployment, many researchers have assumed lower levels of functionality in their cognitive radio. For instance, in his remarks at the 2005 MPRG Technical Symposium, Bruce Fette, Chief Scientist at General Dynamics Decision Systems, noted that many members of the defense community refer to the cognition cycle as the "OODA" loop – emphasizing only the observation, orientation, decision, and action portions cognition cycles. Even the source of the most expansive interpretation of cognitive radio [Mitola_00] suggests that learning would occur during sleep or "prayer" (insight gained from external entities) epochs and that during wake epochs the cognitive radio would primarily operate as an OODA loop augmented by some light planning capabilities. Whether implemented as an ontological cognitive radio or as a procedural cognitive radio, all cognitive radios are likely to make use of a goal driven OODA loop for its adaptations. This assumption of an explicit or implicit OODA loop is reflected in the modeling introduced in Chapter 2.

1.2.1 Radios

The following sections briefly describe some initial cognitive radio implementations and their relationship to the levels of cognitive radio functionality and our classification of cognitive radios. The first two radios discussed – CR1 and DARPA's xG architecture – are examples of ontological reasoning radios. The next two radios – a biologically inspired cognitive radio and CORTEKs – are examples of nondeterministic procedural radios. The last cognitive radio presented – XG1 – is an example of a deterministic procedural radio.

1.2.1.1 CR1

CR1 or Cognitive Radio 1 is the cognitive radio architecture developed by Mitola as part of his dissertation [Mitola_00]. CR1 utilizes case-based and natural language reasoning guided by an OODA loop and an ontological description of the radio's capabilities (**R**adio **K**nowledge **R**epresentation **L**anguage) to determine the adaptations of the radio.

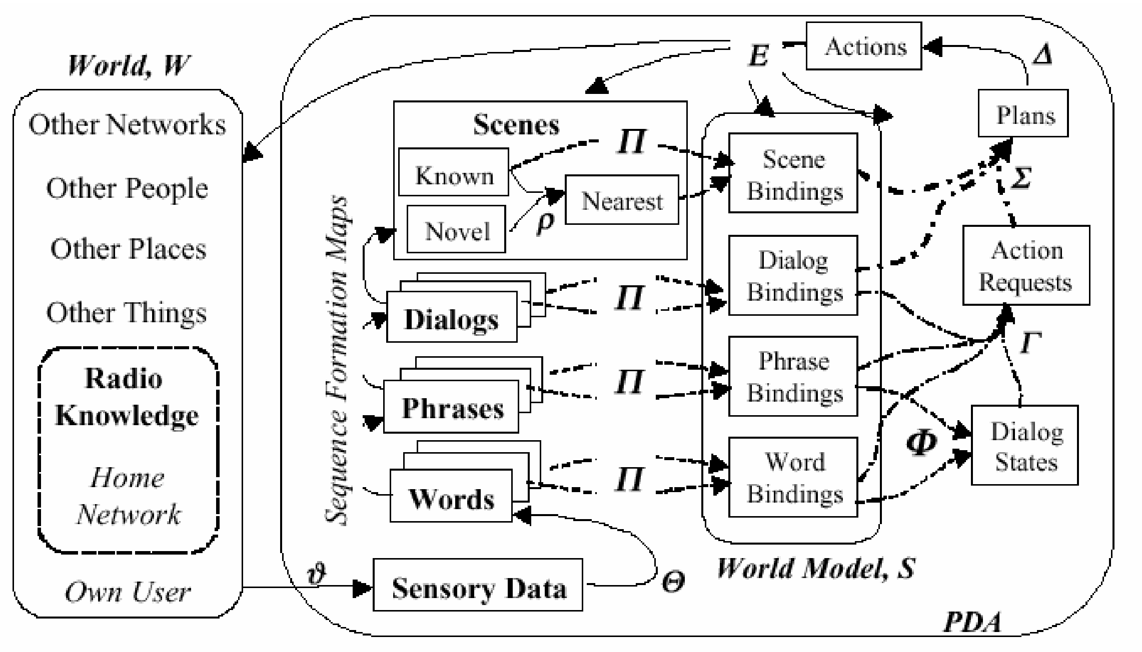

Figure 1.2: CR1 Natural Language Architecture from Figure 8-3 in [Mitola_00]

1.2.1.2 xG

Though hesitant to call its work cognitive radio, DARPA's xG program is pursuing an implementation of cognitive radios that incorporate ontological reasoning into the decision process. A general architecture for their radio is shown in Figure 1.3 where software control is exerted over the radio platform (making the platform an SDR). Note that their radio actually includes two different reasoning engines – one dedicated to policy and one dedicated to waveforms (strategy).

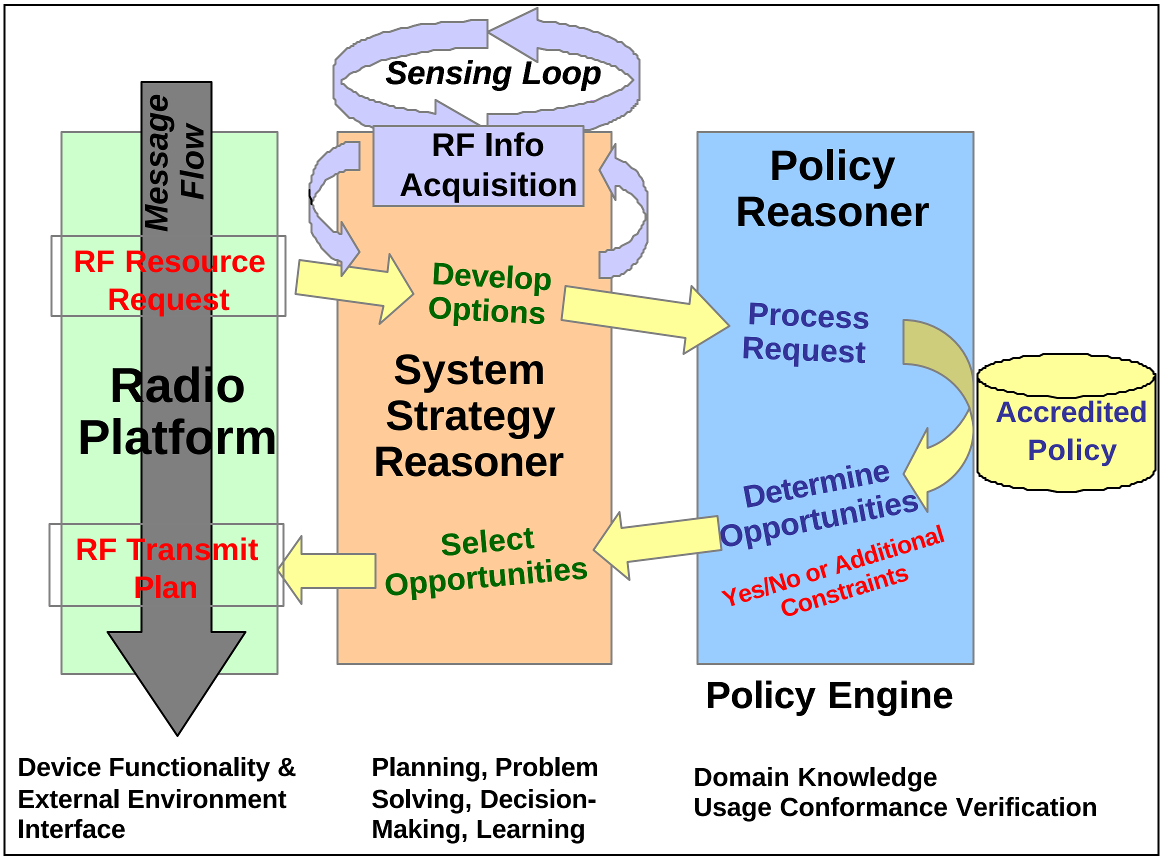

Figure 1.3: DARPA XG High-Level Architecture from [IEEE 1900.1]

1.2.1.3 Biologically Inspired Cognitive Radio

The biologically inspired cognitive radio was the dissertation topic of Christian Rieser [Rieser_04]. Leveraging earlier work on channel sounding, this cognitive radio uses channel measurements to build a hidden Markov model (HMM) of its environment. This HMM is then used by a genetic algorithm to internally predict the performance of different combinations of waveform components for the observed channel conditions.

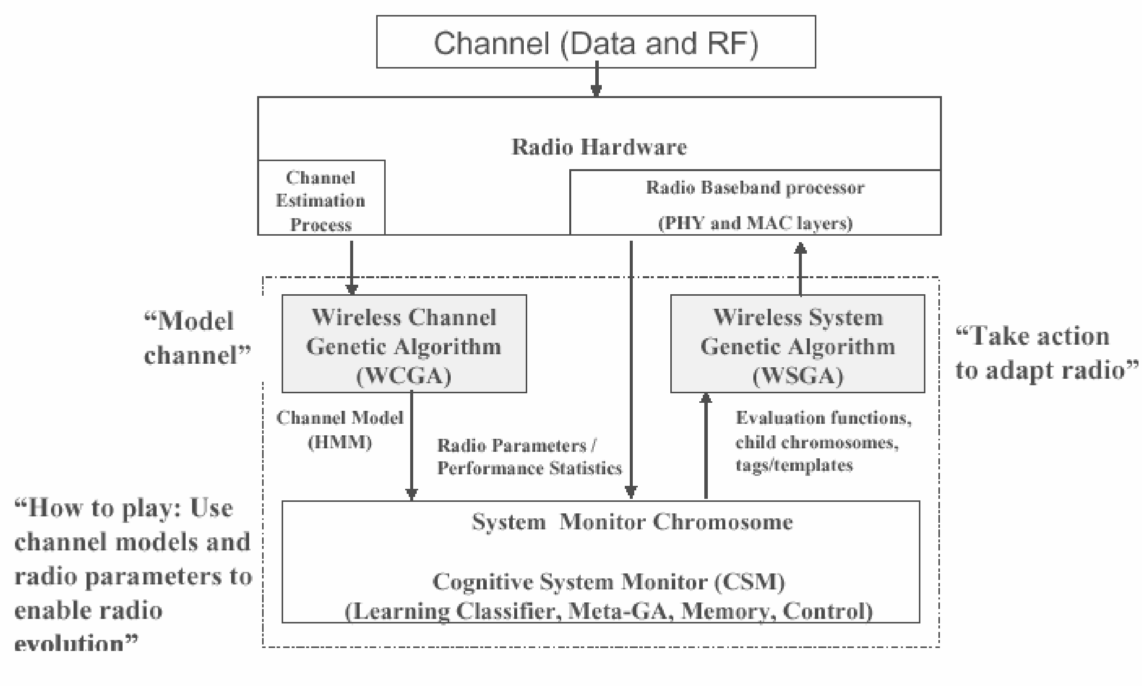

Figure 1.4: Biologically Inspired Cognitive Radio Architecture from Figure 3.4 in [Rieser_04]

Originally intended for use on a Proxim hardware radio, the architecture has since been updated (see Figure 1.5) for use on a software radio. This updated architecture now includes support for policy, classification of signals via neural nets, and user driven inputs. Further, this cognitive engine is intended to be with portable across hardware platforms [Scaparoth_06] and as such the engine has been applied to a GNU[7] radio and to a radio built using Fujitsu test equipment, and will be applied this year to the Innovative Wireless Technologies (IWT) Unified Radio Architecture (URA).

[7] GNU is a recursive acronym which stands for "GNU is Not Unix"

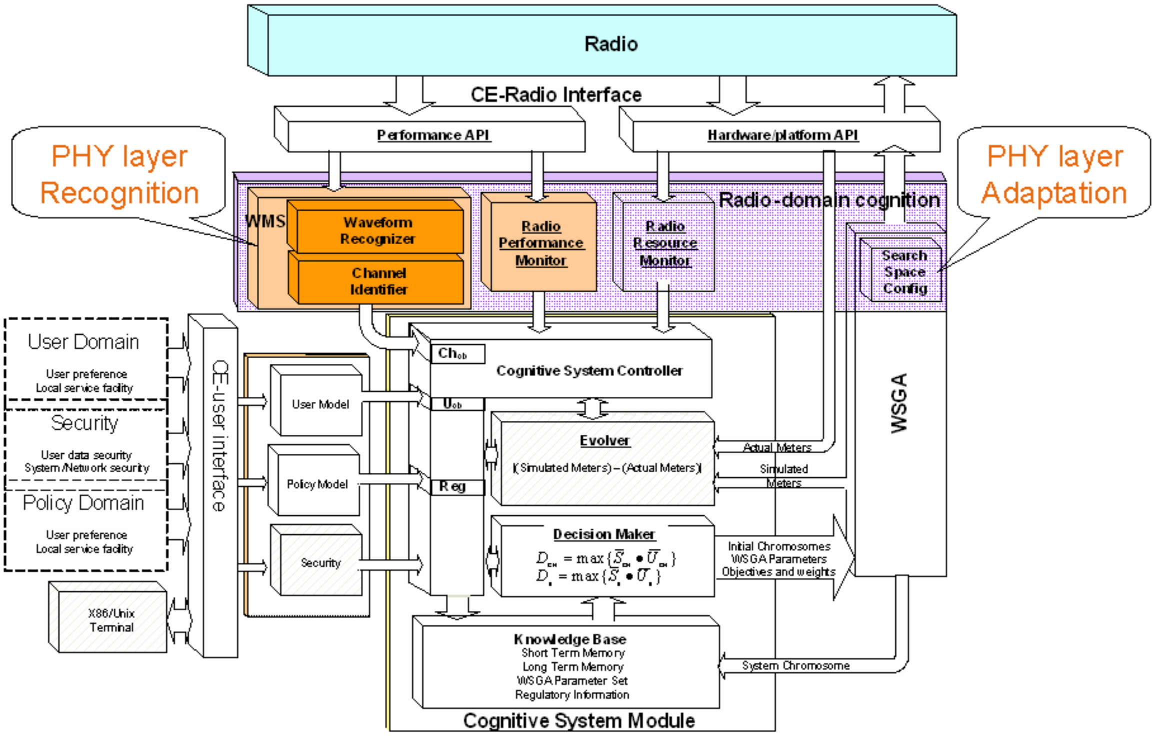

Figure 1.5: Updated Biologically Inspired Cognitive Radio Architecture [Le_06]

1.2.1.4 CORTEKS

The CORTEKS radio is another procedural cognitive radio implemented at Virginia Tech using a PC that leverages Virginia Tech's OSSIE SCA implementation and the following test equipment from Tektronix

- Arbitrary Waveform Generator AWG430 – used to create a multi-mode transmitter
- Logic Analyzer – used for signal characterization (identifying bit patterns, protocols, etc.)
- Real Time Spectrum Analyzer (RSA3408A) – used to perform signal demodulation.

Governed by software defined policy, the CORTEKS radio acts as a secondary spectrum user and adapts its frequency and modulation to maximize goodput while avoiding interference with primary users. To help determine the presence of primary spectrum users, the CORTEKS radio employs neural nets for signal classification.

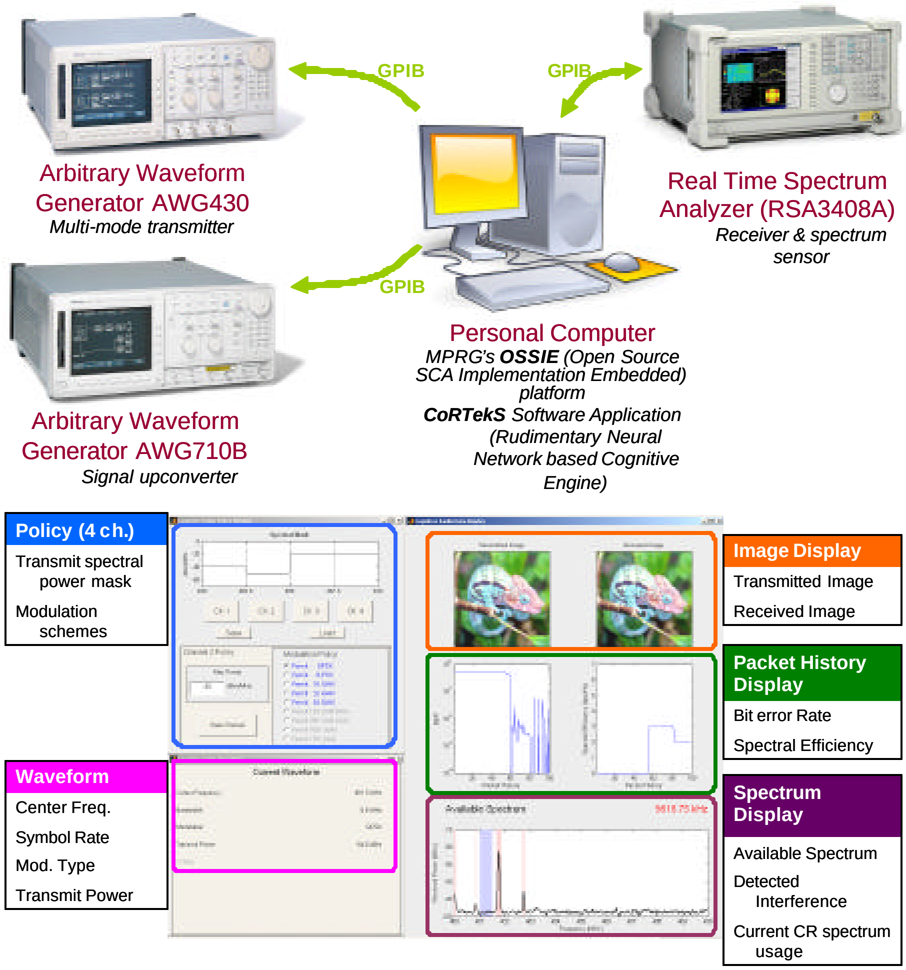

Figure 1.6: CORTEKS Components and Interface

1.2.1.5 Adapt4 XG1

Adapt4 has developed and is currently fielding a cognitive radio called XG1 depicted in Figure 1.7. Intended to operate as secondary spectrum devices, their radio uses a proprietary algorithm known as **A**utomatic **S**pectrum **A**daptation **P**rotocol (ASAP). According to [Adapt4_technology] this algorithm incorporates dynamic frequency selection, frequency hopping, and transmission power control with the intent of avoiding (when possible) and minimizing interference to primary spectrum users.

Figure 1.7: Adapt4's XG1 Cognitive Radio. Image from http://www.adapt4.com/

1.2.2 Cognitive Standards

As highlighted in Section 3, many currently envisioned cognitive radio applications represent "low-hanging-fruit" that could be implemented by incorporating knowledge about the environment and the device into a software radio's control process. Thus it should not be surprising to see that some efforts are already underway to develop cognitive radios with some

1.2.2.1 Policy Radio Deployments

Policy based radios are the logical result of software radio and global mobility. Because of varying historical local needs, different regions of the world implement different sets of regulations. While there has been some movement towards spectrum harmonization, e.g., the push to harmonize the 5 GHz access for unlicensed 802.11, it seems unlikely that all spectral regulations around the world will be harmonized in the near future.

Accordingly, as a radio moves around the world, it requires some mechanism for determining which set of regulations it is operating under. In addition to global phones which are in some sense policy-based radios, though not cognitive policy-based radios, WLAN standards 802.11e and 802.11j can be seen as establishing a protocol that necessitates the use of a policy based radio when operating in the 5GHz band.

A more generalized policy-based radio suitable for cognitive radio is being developed by DARPA under the xG program. As noted in [Berlemann_05], an XML based policy description language has been developed which is loosely based on concepts from game theory. As noted in [Marshall_05b], these declarative policy languages have had significant success with Dynamic Frequency Selection algorithms.

1.2.2.2 Emerging Cognitive Radio Standards and Deployments

The IEEE 802 community is currently developing two standards the directly relate to cognitive radio – 802.22 and 802.11h. Additionally, 802.11k is developing techniques for incorporating radio resource management information into WLAN operation – in effect incorporating knowledge about the environment and the radios.

1.2.2.2.1.1 802.22

There are three applications typically discussed for coexistence with initial trial deployments of cognitive radios: television, microwave point-to-point links, and land mobile radio. Each of these applications has been shown to dramatically underutilize spectrum on average. However, only television signals have the advantage of incumbent signals that are easy to detect (as opposed to a microwave point-to-point links) and not involved in life-critical applications (as would be the case for many land mobile radio systems).

Throughout its history, the UHF bands were under-allocated as regulators underestimated the cost-effectiveness of establishing new TV towers in these bands. It was not until the advent of cable TV that smaller TV stations were capable of cost-effective operation. Now with the introduction of HDTV technology, regulators in the US plan to force a nation-wide switch to this more efficient modulation by 2009 [Rast_05] accompanied by a completion of a de-allocation from analog TV of 108 MHz of high quality spectrum.

With these bands in mind, the 802.22 working group is pursuing the development of a waveform intended to provide high bandwidth access in rural areas using cognitive radio techniques. In a report presented at DySPAN [Cordeiro_05], it is stated that the 802.22 standard intends to achieve spectral efficiencies of up to 3 bits/sec/Hz corresponding to peak download rates at coverage edge at 1.5 Mbps. Simultaneously, the 802.22 system hopes to achieve up to 100 km in coverage.

While the PHY and MAC are still under development, the MAC will provide the cognitive capabilities as it manages access to the physical medium, responsible for quickly vacating a channel as needed. The standard under development has specified the following thresholds for vacating a channel for the following signals:

- Digital TV: -116 dBm over a 6 MHz channel
- Analog TV: -94 dBm at the peak of the NTSC (National Television System Committee) picture carrier
- Wireless microphone: -107 dBm in a 200 kHz bandwidth.

Thus these radios will be required to both detect and classify signals in its environment. To help minimize the interference induced to these signals, the 802.22 protocol is currently considering using spectrum usage tables that will be updated both automatically and by the system operator. To limit the impact when the systems fail to detect the incumbent systems, the standard also places traditional maximum transmission power limits and out-of-band emission limits.

While a promising approach, it is difficult to estimate how wide-scale a deployment 802.22 will enjoy as WiMAX was first to market with deployments in Korea (WiBro) and planned deployments in the US [Segan_06] and will be able to provide the same target service: high data rates to rural users.

1.2.2.2.1.2 802.11h

Unlike 802.22, 802.11h is not formulated as a cognitive radio standard. However, the World Wireless Research Forum [WWRF_04] has noted that a key portion of the 802.11h protocol – dynamic frequency selection – has been termed a "cognitive function". To see why an 802.11h WLAN might be considered a cognitive radio, consider that the 802.11h protocol requires that a WLAN be capable of the following tasks.

- <u>*Observation*</u> – 5.4.4.1 in [802.11h] requires WLANs to estimate channel characteristics such as path loss and link margin and 5.4.4.2 further requires the radios estimate channel characteristics such as path loss and link margin.

- *Orientation* – Based on these observations, the WLAN has to determine if it is operating in the presence of a radar installation, in a bad channel, in band with satellites, or in the presence of other WLANs.
- *Decision* – Based on the situation that the WLAN is encountering, the WLAN has to decide to change its frequency of operation (*Dynamic Frequency Selection*), adjust the transmit power (*Transmit Power Control*), or both.
- *Action* – The WLAN has to then implement this decision.

Reviewing most of the definitions from before, only learning or "recalling and correlating past actions, environments and performance" is not required as part of the standard. However, if we move beyond the requirements of the standard to expected implementations, it seems reasonable that many vendors will include and leverage some memory of past observations (useful for detecting intermittent transmitters) which implies that both cognitive radio definitions will be satisfied.

1.2.3 Institutional Initiatives

Beyond these initial deployments, several entities have started publicly acknowledged initiatives into cognitive radio including DARPA, the SDR Forum, IEEE, and the FCC.

1.2.3.1 DARPA

DARPA sees cognitive radio as a key enabling technology to their vision of advanced networking by allowing less individually capable radios to perform complex operations needed make better use of spectrum and support high data rate applications. Currently, DARPA is exploring many different aspects of cognitive radio as part of the xG program and other ongoing programs. Unfortunately, many of the results of the DARPA programs are not currently in the public domain. However, Preston Marshall, program manager for DARPA's cognitive radio initiatives has promised that contracting organizations will be required to disclose most of their results online in the near future. In the interim, Marshall highlighted many of DARPA's plans and results in a presentation at DySPAN [Marshall_05a] and during a panel session at the SDR Forum [Marshall_05b].

In the area signal classification and detection, DARPA has developed a sensor capable of processing 5 GHz/second frequency capable of sub-noise-floor signal detection (20 dB below) by exploiting cyclostationarity properties. DARPA has contracted with Rockwell to miniaturize this sensor.

Believing that procedural approach would result in too much code and too many detailed policies, DARPA has developed a declarative policy language that is independent of the implementation platform. Already successfully demonstrating small networks of Dynamic Frequency Selection networks, DARPA hopes to extend their policy work to construct a "provable framework" that supports policy enforcement and optimization (current focus is just on making the technology work, the Wireless Networking After Next program is intended to include optimization as a goal). Another demonstration of radios from Lockheed-Martin, Shared Spectrum, and Raytheon with the target of 90% connectivity and 90% chance of finding available spectrum found that 15 times more radios could be fielded using the xG approach. By August 2007, DARPA plans to have field trialed systems of interacting and collaborating 25 xG nodes.

DARPA also believes that advanced network topologies will be a key application of cognitive radios and is starting the CBMANET program to explore advanced networking topologies based on the xG radio. To help support these new dynamic topologies and the proposed optimization routines Marshall believes there may need to be new layers inserted into the protocol stack for topology an optimization because of the intelligence required in those operations.

The most notable anticipated activity from DARPA is the launching of the **W**ireless **A**daptable **N**ode **N**etwork (WANN) project this September.WANN hopes to demonstrate reduced device cost (targeting ~$500/node) via intelligent adaptation and greater node density. Additionally, the WANN program is hoping to achieve significant gains in throughput and network scalability through the incorporation of intelligence in the radios.

1.2.3.2 SDR Forum

The SDR Forum chartered two groups in 2004 to explore cognitive radio issues: the Cognitive Radio Working Group and the Cognitive Radio Special Interest Group. The working group is tasked with standardizing a definition of cognitive radio and identifying the enabling technologies for cognitive radio. The special interest group is tasked with identifying attractive commercial applications of cognitive radio for which the working group should identify the enabling technologies.

At the 2005 SDR Forum Technical Conference, significant emphasis was given to cognitive radio with two paper sessions, one panel session, a tutorial, and a keynote talk dedicated to the subject of cognitive radio. Then in April 2006, the SDR Forum held a Cognitive Radio Workshop in San Francisco.

1.2.3.3 IEEE

The IEEE has expressed significant interest in cognitive radio. As a body, IEEE submitted the proposed definition to the FCC noted in Section 2. To allow for more focused development, the IEEE has started the IEEE 1900 group to study the issue of cognitive radio. Currently, the 1900 group has three subgroups with the following focuses:

- 1900.1 - Standardize definitions and terminology related to cognitive radio
- 1900.2 – Standardizing a process for testing and verifying the operation of cognitive radios.
- 1900.3 – Standardizing approaches for qualifying software modules.
- 1900.a – Regulatory certification of cognitive radios.

The IEEE Communications Society held its first Dynamic Spectrum Access Networks (DySPAN) in November 2005 with a primary focus on how cognitive radio, including the following:

- technologies needed to implement cognitive radio ranging from sensing, analysis of interactions, and advanced networking technologies
- appropriate regulatory approaches for cognitive radio
- potential market opportunities for cognitive radio
- trial implementations of cognitive radio systems.

The IEEE is playing an expanding role in the development of cognitive radio forming the 1900.4 workgroup (joining the 1900.1, 1900.2, and 1900.3 groups described in the December 2005 report) to support standardization of radio regulatory compliance, sponsoring CrownCom2006 – a cognitive radio focused conference – and two special issue journals on cognitive radio, and the 802.22 group which merged its final two proposals in March implying that the standard could be agreed upon within the next year.

1.2.3.4 FCC

On May 19, 2003, the FCC convened a workshop to examine the impact that cognitive radio could have on spectrum utilization and to study the practical regulatory issues that cognitive radio would raise. After a series of public interactions, the FCC adopted the transmitter-centric definition of cognitive radio listed in Section 2 and appears interested in adjusting its regulations in a way that will accommodate the deployment of cognitive radio in unlicensed bands and possibly in a portion of the new bands being opened up by the upcoming UHF reallocation with possible later extensions to the public safety and ISM bands.

Since then, the Federal Communications Commission (FCC) has also taken steps to increase the opportunities for cognitive radio deployment by expanding the unlicensed 5 GHz and requiring that devices operating in those bands support both Dynamic Frequency Selection (DFS) and Transmit Power Control (TPC) – characteristics of 802.11h which was characterized in the December 2005 report as indicative of minimal cognitive radios. In light of the expanded introduction of cognitive radios, the FCC issued proposed rules for compliance testing DFS radios in April [FCC_06a] with comments due on May 15. The results of this process were compiled into a document released on June 30, 2006 that provides a standard for testing DFS and TPC compliance with regard to radar avoidance and minimizing interference with satellites [FCC_06b]. It is expected that these actions will enhance the opportunities for cognitive radio deployments.

1.2.3.5 Other Institutions

Several other institutions are also currently pursuing cognitive radio research including E^2R, Virginia Tech, Winlab, and BWRC.

E^2R is a European initiative into supporting End-to-End Reconfigurability with numerous participating European universities and companies. E^2R is focused primarily focused on incorporating dynamically radio resource management schemes into existing cellular structures to achieve advanced end-user services with efficient utilization of spectrum, equipment and radio resources on multi-standard platforms.

Virginia Tech currently has several significant cognitive radio initiatives. Two different cognitive radio testbeds that leverage test equipment to effect powerful yet easy to implement software radios are under development with a focus on both public safety and commercial interests. These two projects are exploring techniques for enhancing detection and classification capabilities, learning algorithms, knowledge representation and the effect of interaction of cognitive radios. Work is being performed exploring techniques to exploit collaborative radio to improve network performance. Other projects are exploring techniques for analyzing the interactions of cognitive radios, developing environmental awareness maps, and MAC protocols that can trans. Virginia Tech also maintains a publicly accessible cognitive radio wiki as part of its cognitive radio special interest group at http://support.mprg.org/dokuwiki/doku.php?id=cognitive_radio. Additionally, Virginia Tech is organizing the MANIAC (**M**obile **A**d Hoc **N**etworking **I**nteroperability **A**nd **C**ooperation) challenge wherein researchers from numerous universities will independently design cognitive radios which will then be brought together to "compete" to see which cognitive radio algorithms yield desirable network behavior. A similar competitive cooperative contest is in the planning for DySPAN 2007 though details are unclear at this point.

Winlab at Rutgers University is developing a cognitive radio testbed for disaster response using commercially available components. BWRC is currently developing a cognitive radio for sensing and opportunistically using the spectrum. Additionally, BRWC is researching techniques for improving spectrum sensing algorithms.

1.3 Cognitive Radio Applications

Applications are often included in the definition of cognitive radio because of the compelling and unique applications afforded by cognitive radio. Additionally, there are many existing SDR techniques that cognitive radio is expected to enhance. This section reviews the following frequently advocated applications of cognitive radio some of which will be used as inspiration for example analyses later in this document:

- Improving spectrum utilization & efficiency
- Improving link reliability
- Less expensive radios
- Advanced network topologies
- Enhancing SDR techniques
- Automated radio resource management.

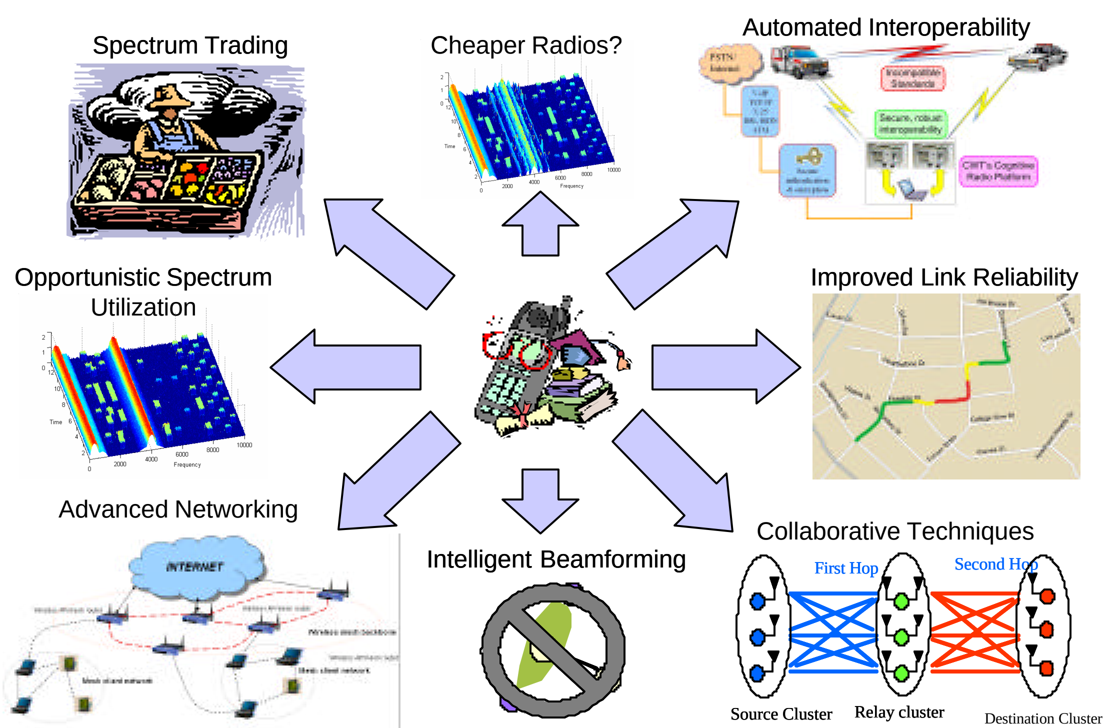

Figure 1.8: Cognitive Radio Applications

1.3.1 *Improving spectrum utilization and efficiency*

Wireless technologies and wireless devices have proliferated over past decade dramatically increasing the demand for electromagnetic spectrum. Because of the current approach to spectrum access, spectrum supply has not kept up with spectrum demand leading to the appearance of scarcity in the electromagnetic spectrum.

However, research performed by various entities such as the FCC indicates that this assumption is far from reality; there is available spectrum since most of the spectrum allocated sits underutilized. In a recently completed NSF funded study of allocated spectrum utilization, researchers at Kansas University found an average U.S. spectrum occupancy of 5.2% with a maximum occupancy of 13.2% in New York City. Figure 1.9 shows the specific measurements by band as averaged over the following six locations: 1. Riverbend Park, Great Falls, VA, 2. Tysons Corner, VA, 3. NSF Roof, Arlington, VA, 4. New York City, NY, 5. NRAO, Greenbank, WV, 6. SSC Roof, Vienna, VA [McHenry_05].

So while the dramatically increasing demand for spectrum has fostered a perception that spectrum is scarce, the reality is that spectrum is abundant but poorly utilized.

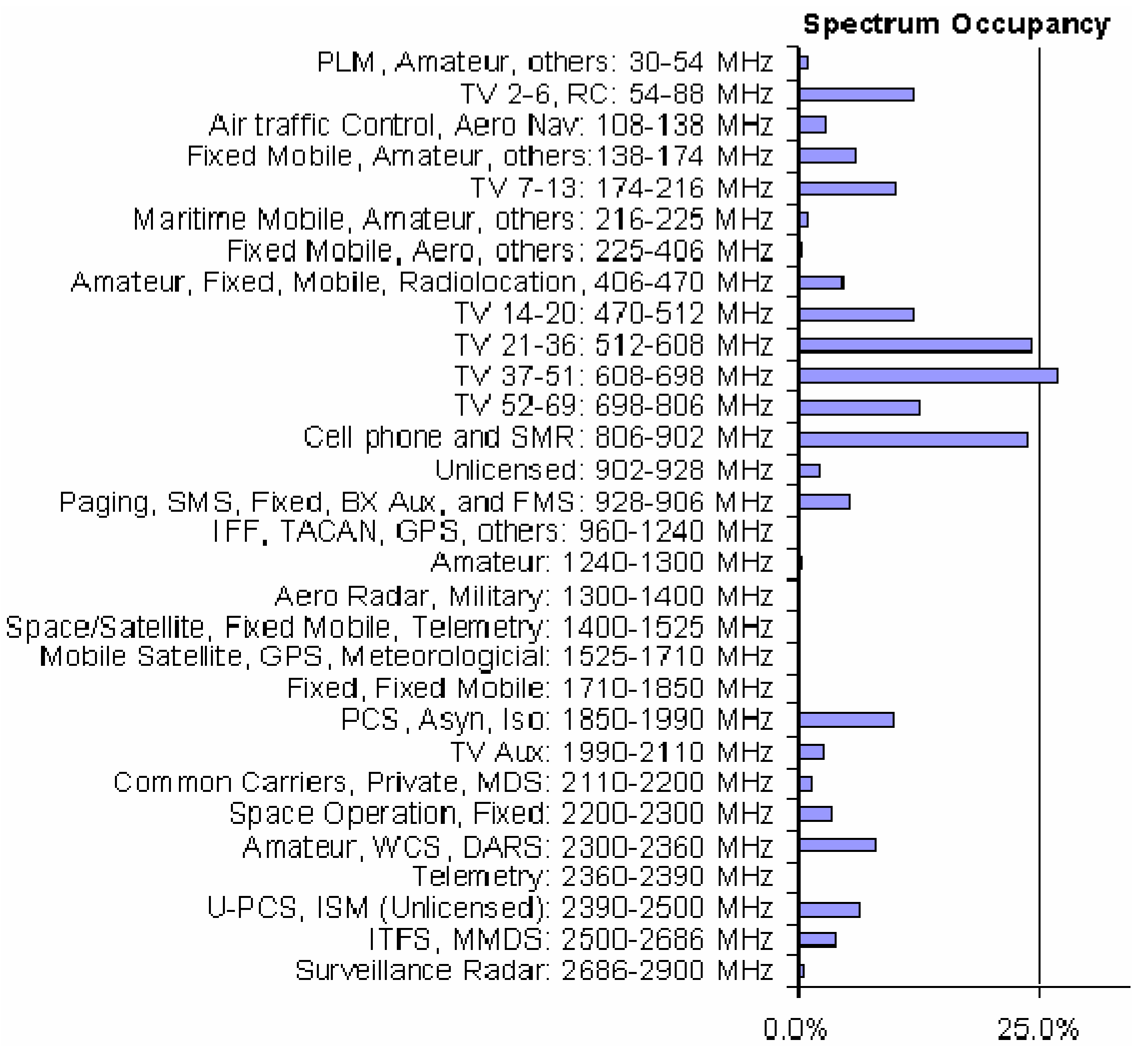

Figure 1.9: Spectrum availability by band. Adapted from Figure 1 in [McHenry_05].

This underutilization is the result of a number of different factors including overly conservative allocation of guard bands; a migration from spectrally inefficient analog waveforms to more efficient digital waveforms, and the natural gaps in utilization that occur throughout the day due to variations in demand. As an example of variations in demand, Figure 1.10 shows a Matlab depiction of spectrum measurements made in Germany at Karlsrhue in a more heavily used band. As the figure illustrates there is significant variation in spectrum underutilization in time and frequency, and though not depicted, there is also significant variation in terms of location.

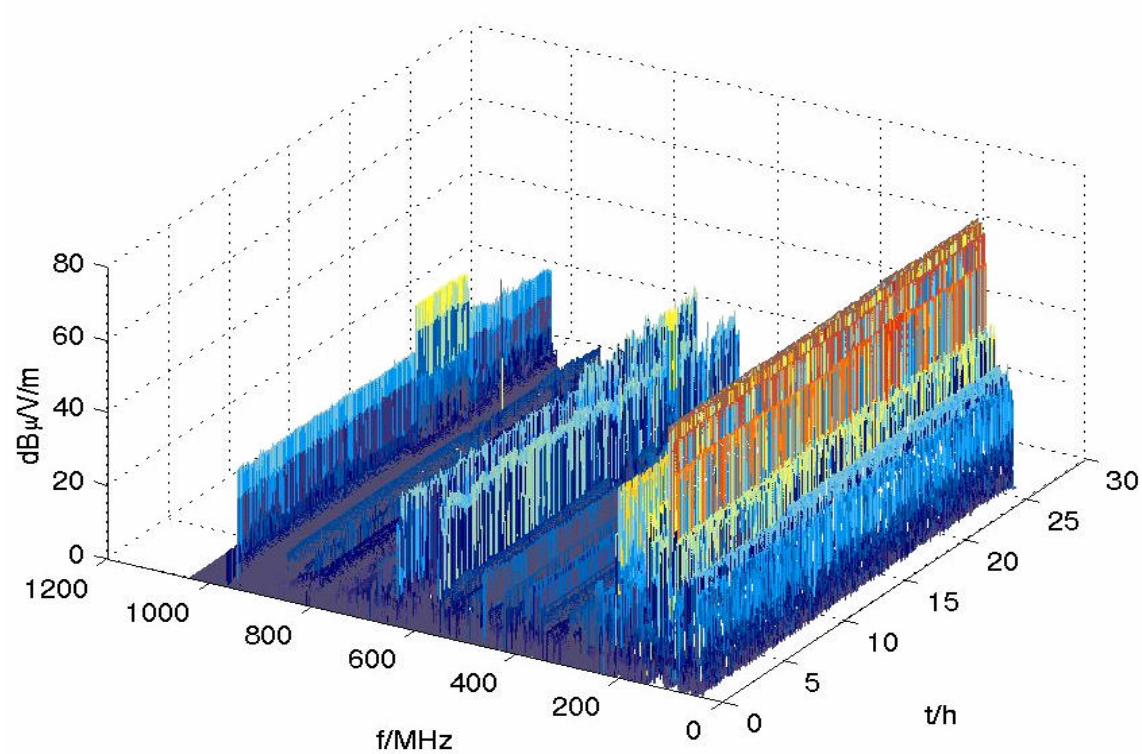

Figure 1.10: Matlab capture of channel measurements from Germany [Jondral_04]

To improve spectrum utilization, *opportunistic spectrum utilization* has been proposed wherein devices occupy spectrum that has been left vacant. An illustrative example of opportunistic spectrum utilization is shown in

Figure 1.11. In the left half of the figure, a pair of transmitted carrier signals is present in the lower frequency bands while a random access system and a TDMA system are operating in the upper bands. After observing the *spectrum holes* - points in time and frequency where spectrum is underutilized – opportunistic devices could fill in these holes to support concurrent services as illustrated in the diagram on the right.

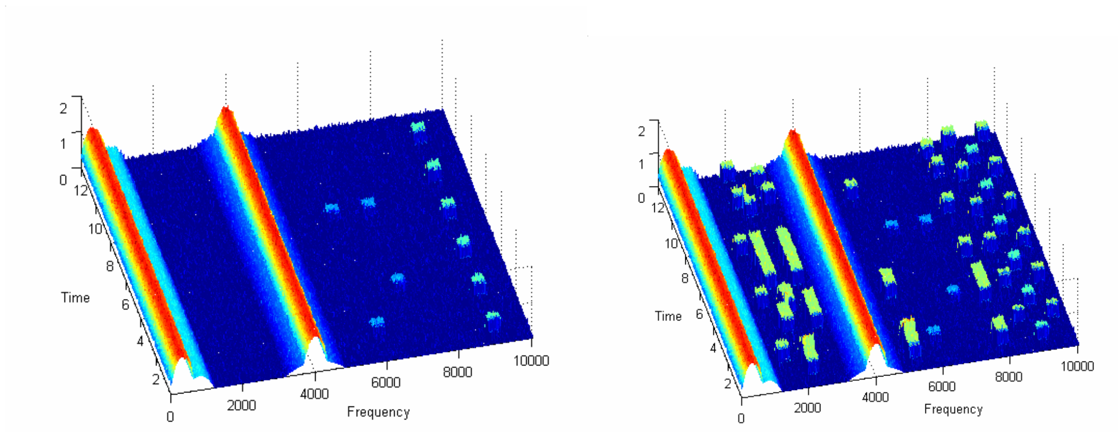

Figure 1.11: Conceptual example of opportunistic spectrum utilization

According to the xG program manager [Marshall_05a], cognitive radios that employ opportunistic spectrum utilization have been shown to provide a 10-fold gain in capacity by implementing dynamic frequency selection algorithms.

Of course, opportunistic use of spectrum presents significant challenges to the technical and regulatory communities. From a technical perspective, the devices must be able to autonomously resolve conflicts over spectrum access and when operating opportunistically should be able to avoid interfering with incumbent signals. While avoiding introducing interference to a signal that is continuously present, it is more difficult with regularly structured, but intermittent signals such as a TDMA signal, and impossible to guarantee with a random access signal. To some extent this implies that the control processes of cognitive radios will need to be able to operate over multiple time scales as is proposed in [Moessner_05].

Since there is a technical problem, there is also a regulatory problem. Current spectrum licensees are generally only amenable to opportunistic spectrum access when they can be assured that their signals will not be degraded. Figuring out how to achieve the 10-fold gain in capacity while limiting the impact of existing services is currently a topic of much debate in the regulatory community.

1.3.2 Improving Link Reliability

After improving spectrum utilization, the second most commonly discussed application of cognitive radio is improving link reliability. Many adaptive radios currently improve link reliability by adapting transmission power levels, modulations or error correction. However, a cognitive radio that is capable of remembering and learning from its past experiences can go beyond these simple adaptations as can be shown via the following simple example.

Figure 1.12 illustrates a path that a mobile subscriber might follow on his daily commute through a particular area of a city where signal quality usually drops to an unacceptable level (shown in red) due to a coverage gap. Perhaps the first time or perhaps after several occurrences, the cognitive radio would become aware of this problem. Then via some geo-locational capability or by learning the expected time of day when this occurs, the radio could anticipate the coverage gap and signal to the base station the need to alter the signal characteristics as the user approaches the coverage gap.

Figure 1.12 Path and associated signal quality for a cognitive radio.

The same concept of detecting coverage gaps could also be employed at the base station where the base station would learn to correlate particular areas of its coverage area with a gap and then could adjust its operation (perhaps via beam forming) to eliminate the gap. Without including a cognitive base station, cognitive mobiles could share such information among themselves so that the mobiles may learn to improve their link performance without first experiencing the coverage gap. This, however, highlights another key challenge to realizing cognitive radios – how to represent the knowledge a cognitive radio needs to operate in a machine-usable and machine-to-machine translatable way.

1.3.3 Less Expensive Radios

While adding complexity to a radio's control processes would appear at first glance to necessarily increase cost, the inclusion of a cognitive control process may significantly decrease device cost when cognition is enabled. To resolve this apparent paradox of adding features but reducing cost, it is important to note that many of the proposed applications of cognitive radios represent "low-hanging fruit" that can be implemented via low complexity control processes. Further, these cognitive processes would be implemented in a software defined control process for which additional computations are relatively insignificant, especially when compared to the cost of improving the performance of analog components. Adding a couple hundred software cycles per second is virtually costless; improving the performance of a RF front end by 3 dB can be a very expensive undertaking.

As noted in the preceding, the inclusion of opportunistic spectrum utilization permits significant gains in terms of capacity. Instead of only improving capacity, some of the spectrum gain could be "given" to accommodating lower performance analog components in the transmitter which generally result in signal energy outside of the intended band. These lower performance transmitter analog components can be included in the cognitive radios or among "dumb" radios.

For example consider the spectrum utilization diagram shown below in Figure 1.13 where the signal from one device exhibits significant spurious components and the remaining cognitive devices are capable of observing this signal and adapting around these spurs. In this particular example, there would be no degradation in total informational throughout bandwidth when compared with the example considered in Figure 1.11 as all opportunistic devices are still capable of finding spectrum holes to transmit in. However, any degradation in terms of out-of-band energy necessarily decreases the available bandwidth for opportunistic spectrum utilization so some tradeoff has to be made.

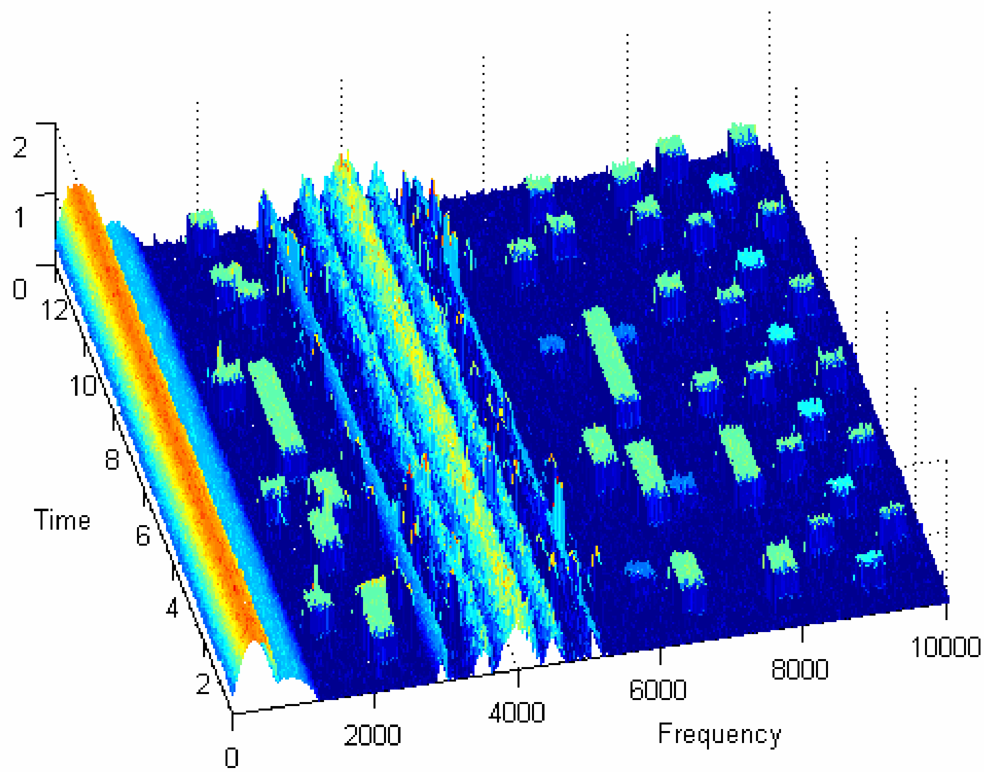

Figure 1.13: Opportunistic spectrum utilization in the presence of device with significant signal degradation.

If the lower performance analog components are present only at the receiver, then there is no direct effect on the observable spectrum. However, cognitive radio processes similar to those assumed necessary for ensuring link reliability can be applied to overcome the limitations of poorly performing analog front ends. For example, adaptive beam forming or nulling can provide additional SINR or opportunistic spectrum utilization routines could seek "deeper" spectrum holes to overcome low-Q anti-aliasing filters.

Whether included in transmitter or the receiver, cognitive radio facilitates the use of lower cost analog components. Of course, these gains can be supplemented by software radio techniques such as dithering data converter inputs and predistortion for power amplifiers.

1.3.4 Advanced network topologies

Under a MANET operational scenario, the access points or base stations do not have to maintain direct connections to the more distant regions of their clusters or cells. Instead, each base station only needs to be able to reach a handful of the closest subscribers while the devices farther from the base station gain access by communicating through a sequence of intermediate devices to reach the base station. As illustrated in Figure 1.14, in a MANET the average propagation distance for each link is much shorter than would be the case for a star topology with the same number of base stations. The shorter propagation lengths means that greater effective spectral reuse factors can be achieved which some have said would lead to a gain of up to 30 dB in system capacity [Fette_05].

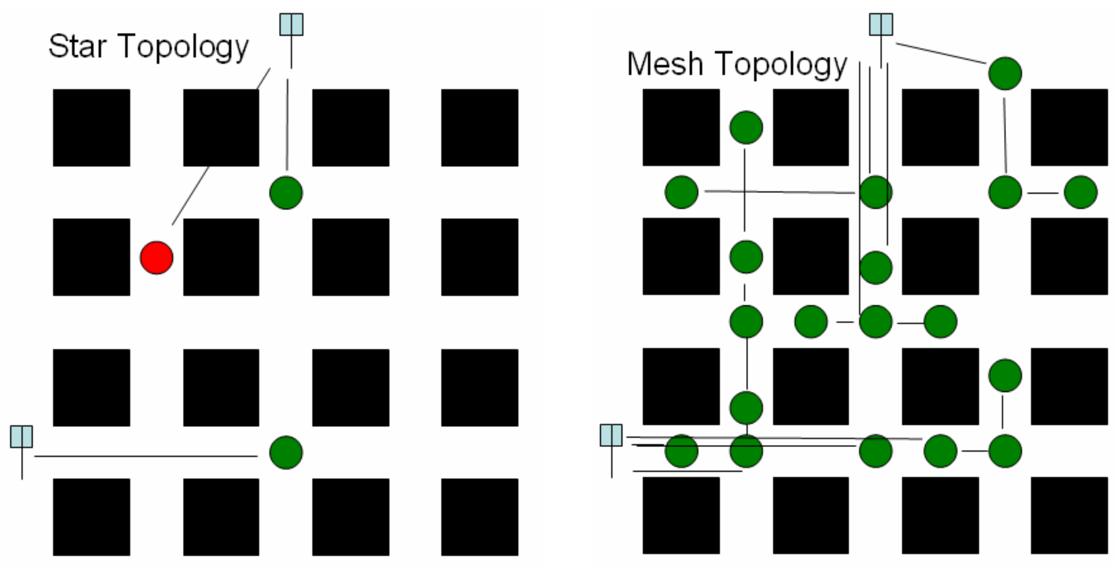

Figure 1.14: Star and Ad-hoc Topologies

While the deployment and advantages of MANETs do not inherently require the use of cognitive radio, cognitive radio can be seen as an enabling technology. For a MANET to successfully operate two criteria should be satisfied. First, a high node density should be present to permit the use of lower power links. In general, the denser the network of devices is, the greater the theoretical capacity of the MANET. Second, the devices must be capable of supporting the dynamic routing and link maintenance routines required ensure network connectivity.

As described previously in this document, cognitive radios can be used to significantly increase the usable bandwidth and decrease device cost which in turn implies that many more devices can be expected to be fielded in the future, thus implying greater device density. For the second criterion, the environmental and device awareness implicit to a cognitive radio facilitate implementation of the algorithms needed to support the MANET routing and link maintenance algorithms.

1.3.5 Collaborative Techniques

A collaborative radio is a radio that leverages the services of other radios to further its goals or the goals of the networks. As introduced in the previous section, collaborative radio can be viewed as an application of cognitive radio. However, a collaborative radio could be implemented without a full implementation of cognitive radio. For instance, many collaborative applications require only trivial learning processes. Nonetheless cognitive radio can be viewed as an enabler of collaborative radio in that cognitive processes simplifies the identification of potential collaborators and intelligent observation processes facilitates the inclusion of distributed sensing – a characteristic of many collaborative radio applications.

One of the more frequently discussed ways in which radios can collaborate is by implementing relay channels. In a relay channel, a radio serves as an intermediate node in the path between the client device and the access node. In general, this relaying process can be implemented at the relay node by amplifying and forwarding the received signal or by decoding and forwarding the signal. In the former case, radio complexity is relatively low as the signal does not have to be received; in the latter, radio complexity is generally much higher as the relay has to completely receive the transmitted signal. However, the added complexity incurred by a decode-and-forward approach is generally accompanied by improved performance (low latency waveforms being the most noticeable exception) so there exists a tradeoff between the two approaches.

The concept of using relay radios is currently the focus of the 802.16j workgroup which considers three types of relays: fixed relays, nomadic relays, and mobile relays. As illustrated in Figure 1.15, the relay radios in 802.16j are intended to extend the coverage of 802.16 networks where the relay nodes are intended as an extension of the 802.16 infrastructure.

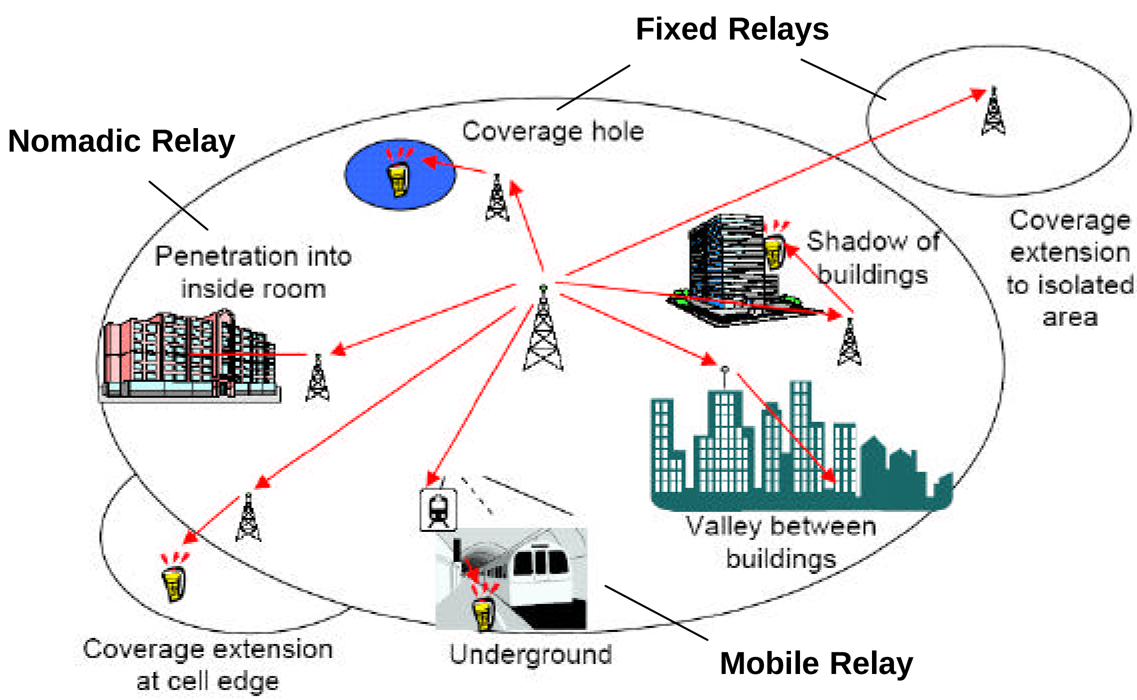

Figure 1.15: Conceptual operation of 802.16j Modified from Fig 1 in IEEE 802.16mmr-05/032

While the relays in 802.16j are dedicated infrastructure installations, the existence of mobile relays (intended to support mass transportation) implies that the relaying concept should be extendable to subscriber units. While relaying with subscriber units implies that performance may be more difficult to guarantee, it should be possible to improve overall network performance and coverage with less deployment costs by judicious choice of relay nodes. However, wisely choosing which subscriber units should act as relay nodes implies some knowledge of the state of the network and the traffic and mobility characteristics of other subscriber units in the area. For a traditional radio, this knowledge would be difficult to come by, but if the subscribers were cognitive radios then presumably the radios would be gathering and processing the relevant information as part of the normal processing.

1.3.5.1 Distributed Antenna Arrays

Of course there will be situations where a group of subscribers is out of range of an access node and no subscriber device will be positioned well enough to serve as a relay node. However, if the subscriber devices collaborate, their effective range can be dramatically increased, perhaps far enough to reach an access point.

In this form of collaboration, several radios collaborate to realize an antenna array thereby leveraging the processing gains of an antenna array without each subscriber unit needing to have its own antenna array. Because of the likely spacing of devices, it seems unlikely that beamforming will be a readily used application for a collaborative array of radios, but diversity applications should be usable. For instance, two different diversity-based collaborative antenna applications are illustrated in Figure 1.16. In a simple diversity scheme a number of radios can coordinate to transmit or receiver the same signal thereby realizing a transmit or receive diversity algorithm. With some additional coordination, those same collaborating radios could implement a MIMO, MISO, or SIMO algorithm depending on the operational context.

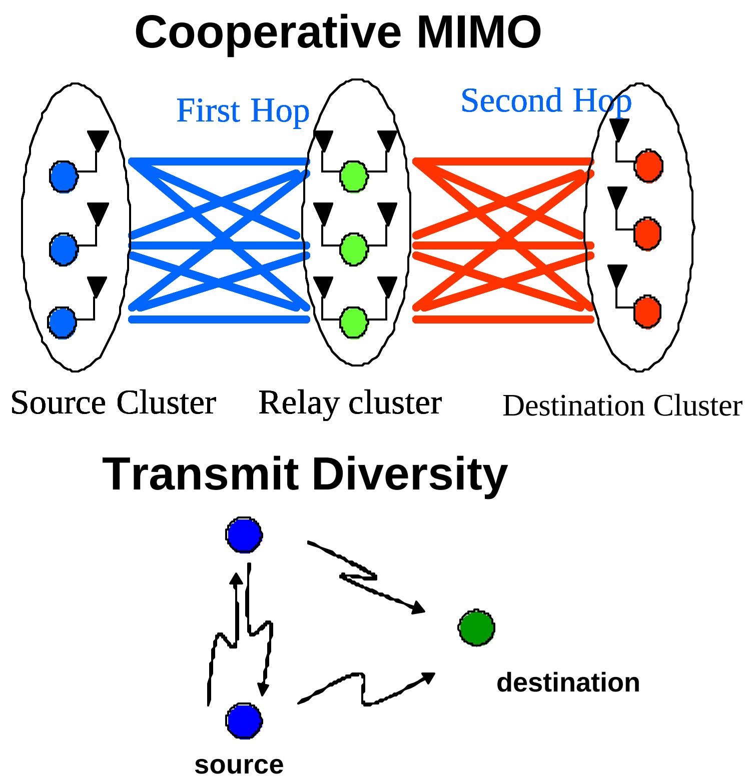

Figure 1.16: Distributed Antenna Array Possibilities

Particularly for the distributed MIMO/MISO/SIMO schemes and to a lesser extent the collaborative transmit and receive algorithms, good timing and localization information will be of significant aid to the performance of these algorithms. Again, the normal processes of cognitive radio may be able to provide the information necessary or perhaps this information could be collaboratively collected into some environmental map.

1.3.5.2 Distributed Mapping

Assuming radios are aware of their location and capable of making observations, it should be possible for radios to collaborate to build maps of their environment. One such map could be the radio environment map discussed in [Zhao_06] which can help inform cognitive radio adaptations. Maps targeted to the service providers and subscribers instead of the radios themselves could also be built via radio collaboration.

For instance, by collecting signal strength measurements and feeding this information back to the infrastructure, a network's coverage map can be built. With this continually updated coverage map, service providers can quickly identify coverage holes and take steps to rectify the problems. Assuming the network infrastructure is implemented using cognitive radio technology, these coverage holes could be automatically filled, significantly decreasing the chances of a subscriber experiencing a dropped call and improving subscriber perception of service.

As another example, suppose the mobiles are continuously returning their location information to the network's base stations. By integrating this information, the network can get an accurate picture of its subscriber density by location. While a subscriber density map will be useful for network planning, it also implies a unique subscriber service – real time traffic maps and real time traffic updates. Specifically, when higher subscriber densities are located on roads, this should be indicative of higher density automobile traffic – information which other drivers may be willing to pay so as to avoid traffic jams. Thus by simply collecting location information from each of its subscribers, a service provider can provide a novel service of real time traffic updates.

1.3.5.3 Enhanced Security

Certain radios will tend to be used in close proximity with other radios. For instance, the various Bluetooth devices in an automobile will typically be operated with the mobile (or mobiles) of its owners onboard. By learning and recognizing the MAC addresses of the mobiles of an automobile's owners, the automobile should be able to flag situations that are inconsistent with normal operation, for instance if the car was in operation and a different mobile than the owners' mobiles was on board. In and of itself, this situation will not be sufficient to know that the automobile has been stolen, but it should be enough to make the situation a scenario worth further examination. Conceptually, this could be viewed as similar to the process wherein credit card companies flag purchases that do not fall into normal patterns. Similarly, this sort of information could be incorporated into an enhanced authentication system where contextual information gleaned from other authorized radios can provide degrees of authentication assuredness.

1.3.5.4 Collaborative Sensing

For many emerging wireless standards, such as 802.22, it will be important for radios to be able to detect and classify signals in its environment to ensure proper network behavior. Introductory statistics courses teach that an increasing number of independent (and unbiased) observations reduce the variance of estimated parameters. Thus the decisions as to if a signal is present and what kind of signal is present (for example is a TV broadcast present or more generally is the incumbent user transmitting) could be improved by incorporating more observations from other devices. Beyond 802.22 applications, collaborative sensing should be able to help mitigate the hidden node problem endemic to most standards.

In fact, collaborative radio itself holds the potential for numerous applications, including relay channels, distributed antenna arrays, improved localization algorithms, and collaborative mapping. However, many of these algorithms lack agreed upon models and algorithms. Without some unified approach, collaborative radios will likely go the way of networking and lack a sound theoretical basis. Lacking this theoretical basis, it will be important to construct prototypes and demonstration system before implementation or standardization can occur.

1.3.6 SDR techniques enhanced by cognitive radio

Similar to how cognitive radio will hasten the wide scale deployment of MANETs without being a requisite technology, several other techniques that require a software radio can be significantly enhanced by the use of cognitive radio. These SDR techniques include antenna array algorithms, spectrum trading, and interoperability.

Smart antenna technology is a traditionally discussed advantage of software radio. However, network performance can be greatly improved by adding environmental awareness to smart antenna algorithms. For example consider the beamforming example shown in Figure 1.17 where two links are present – one between the gray nodes and one between the white nodes. When the bottom left node chooses to implement transmit beamforming, a significant gain in performance for the gray nodes' link can be expected.

However, from a network perspective, this choice is not desirable as the benefit accrued by the beamforming link will not be as great as the added interference that the intermediate white node will experience. However, if the gray nodes are cognizant that one of the white nodes is operating within the potential beam, then the gray nodes could choose a different adaptation that would not impact the white nodes, perhaps via a combination of spatial and frequency multiplexing.

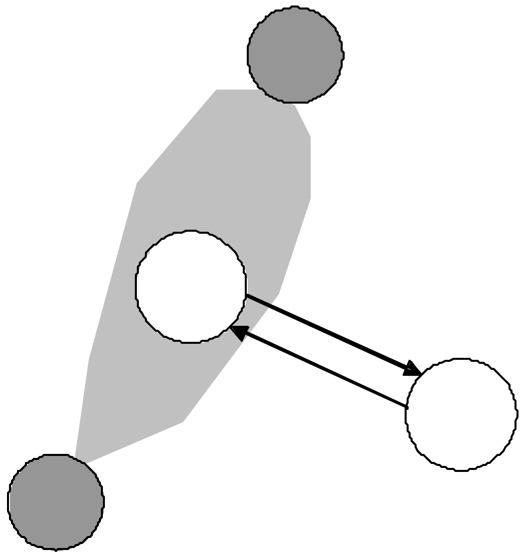

Figure 1.17: An example of ad-hoc beam forming that would have negative effects on network performance.

Spectrum trading has long been discussed as a potential benefit of the frequency agility of software radio. In spectrum trading, different spectrum owners purchase and sell spectrum to varying service providers in response to changes in market demand. In theory – the practice of spectrum trading is in its infancy having recently received limited approval in the UK and Guatemala [Hatfield_05] and FCC approval in the US for trading among public safety users – spectrum trading facilitates the allocation of spectrum in the most efficient manner in terms of demand.

Fundamentally, the only technology required to support spectrum trading is software radio. With software radio, subscriber nodes can be instructed to change their band of operation following any spectrum trade. However, this implies spectrum trading at the service provider level which implies a process that requires weeks to months to complete. However, if each subscriber unit is capable of determining its own bandwidth requirements, is aware of its environment and the availability of spectrum, and is capable of negotiating with service providers for bandwidth, then spectrum trading transactions could be conducted on the order of milliseconds for significantly smaller pieces of spectrum. Similarly cognitive base stations operated by service providers could quickly and dynamically shift spectral resources between providers to adjust to variations in spectral demand, significantly reducing the probability of a dropped or blocked call.

Interoperability is another frequently touted benefit of the reconfigurability of software radios. Assuming perfect reconfigurability, a software radio can be readily reprogrammed to communicate using any waveform necessary to communicate with another radio, whether the second radio is a software radio or a legacy radio. One commonly discussed technique for supporting interoperability among different legacy systems is to utilize one radio as a gateway device and automatically retransmit messages using the waveforms that each legacy device understands.

Elided in this discussion are the control processes that translate device reconfigurability into interoperability. With a software radio acting as the gateway, it is necessary for a network administrator to set up the connections between disparate legacy devices as the gateway node may have no idea of what devices are present or what connections need the services of the gateway. If a cognitive radio serves as the gateway, the cognitive radio can assume responsibility for these tasks in an automated fashion.

1.3.7 Automated Radio Resource Management

After a wireless network is deployed, wireless engineers typically spend a few weeks tuning the radio parameters to get the most out of a network. Channels allocations between sectors, call drop thresholds, power levels, timers, antenna patterns and many more parameters are all adjusted to improve network performance based on post-deployment measurements. With the increasing number of wireless networks and the movement from centralized service providers to home and office wireless LANs, the need to optimize wireless networks will become an increasingly important but will be impractical to be performed at home or in rapidly deployed networks. For instance, Virginia Tech spent months carefully planning and checking up on the deployment of its wireless LAN in order to maximize coverage with an acceptable capacity level – an unacceptable amount of time in a disaster response scenario.

Because of its capacity to observe and learn how to improve its performance, cognitive radio networks could take over the task of post-deployment tuning and automatically update the radio parameters to best suit the needs of the particular deployment. Such an application would have a significant impact on rapidly deployed networks where emphasis, in home WLANs (which are rarely tuned), and in fixed commercial infrastructure where cognitive radio should be able to reduce the demand for post-deployment engineering.

1.4 Key Issues to Wide-Scale Deployment of Cognitive Radios

Of course there are always significant challenges accommodating revolutionary changes. First, unleashing the revolutionary changes of cognitive radio demands the development of new regulatory ideas – traditionally a glacial process. Second, programming intelligence has always been a difficult undertaking, and for the first time intelligence needs to be included in the radio. Third, many cognitive radio applications assume advanced capabilities to detect and classify signals and identify unused spectrum in a timely manner – capabilities that still need improvement. Fourth, to the extent that cognitive radio is an evolved software radio, cognitive radio will also benefit from enhanced control over the hardware, increased processing power, smaller form factors, and improved software verification techniques. Finally, autonomous adaptations of cognitive radios lead to complex interactive decision processes that make performance guarantees and network planning difficult.

Before deploying cognitive radios in a wide-scale manner, there are a number of issues that should be addressed. These include being able to predict how the interactions of cognitive radios influence network performance, addressing regulatory issues, improving environmental observational capabilities, and a number of SDR issues that are exacerbated by cognitive radio.

1.4.1 Regulatory Issues

Partially caused by the present uncertainty in predicting the effect of interacting cognitive radios and partially caused by ideological differences, how to regulate cognitive radios has emerged as a significant point of disagreement, dividing the policy community into two camps: property-rights and commons. While both camps agree that the traditional command and control model wherein spectrum is licensed for a particular application is less desirable and on the way out, there is little agreement between the two models how cognitive radios will be governed in the future.

Under the commons model (also called the unlicensed model), a pure opportunistic usage approach would be adopted wherein a cognitive radio could make use of any available spectrum that it observed. Under the property rights model (also called the exclusive-use model), entities would "own" their spectrum instead of licensing it, thus entitling them to implement different waveforms as well as subdivide their spectrum for resale to secondary spectrum users for a variety of applications including opportunistic spectrum use.

Property-rights proponents claim that the commons model will lead to a tragedy of the commons. A tragedy of the commons is a situation that can occur with a publicly-held finite resource where each person assigns receives a positive benefit from using more of the resource leading to overuse of the resource to the point of catastrophic results. Commons proponents are quick to point out that spectrum is an infinitely renewable resource so we will never run out and thus cannot experience a tragedy of the commons. However, property rights proponents respond that per unit time, spectrum is indeed finite and many apparently boundless resources have been overused when regulated with a commons approach.

Commons advocates claim that a property-rights model may limit the development of technology and could lead to a tragedy of the anti-commons wherein a small number of entities secure the rights to spectrum and exclude others from using the spectrum thus increasing the value of their spectrum and leading to underutilization of the spectrum.

However, property-rights advocates note that spectrum presumably would not be used any worse than it is now and anti-trust laws exist for handling such an anti-commons situation.

While the property-rights approach appears to have the better theoretical argument and while many incumbent service providers have explicitly stated their opposition to the commons model [Lynch_05], it is difficult to argue with the success of 802.11 which was deployed under a commons regulatory scheme in the ISM bands.

While there are significant differences between the two camps, as noted in the remarks of Andy Mudar [Mudar_05], both communities have expressed interest in a simple regulation that could ensure proper and predictable operation of cognitive radios. However, no such regulation has yet to be identified.

1.4.2 Knowledge Representation

The capability to intelligently reason about the environment implies the existence of some language that captures the knowledge that the radio has about the environment. The need for such a language formed a significant portion of the discussion in the dissertation that proposed cognitive radio. Specifically, [Mitola_00] proposed the use of a Radio Knowledge Representation Language (RKRL) to describe the knowledge a radio may have about its own capabilities and its environment.

Similarly the xG program has developed an XML-based language for representing in a declarative manner the policies that govern a cognitive radio's actions [Berlemann_05]. In remarks at a cognitive radio panel discussion at the 2005 SDR Forum, Preston Marshall noted that this declarative language approach had shown significant success with Dynamic Frequency Selection (DFS) algorithms. However, at that same panel concern was expressed over how to validate an debug a radio whose operation is determined by a declarative language, such as Prolog, as opposed to a traditional procedural language, such as C. OWL – Web-based Ontology Language – has also been proposed as a language for representing radio knowledge in a declarative manner, but primarily for the purpose of supporting knowledge queries between radios [Baclawski_05].

Taking an entirely different though potentially complementary route, [Mohammed_05] has shown that significant amounts of information related to cellular channels can be collected and represented using a hidden Markov models (HMM). Further, these HMMs can be used to gain context and environmental awareness by correlating HMMs generated from run-time observations with

At this point, it is uncertain how these languages will interoperate and if the combination will provide a sufficient basis for implementing the reasoning capabilities needed for cognitive radio. Once these knowledge representation languages have crystalized, additional work is expected to be performed in the area of artificial intelligence (AI), e.g., inference machines, which will further enhance the capabilities and advantages of cognitive radio. Fortunately, however, cognitive radio does not require fully operation AI for any of the applications discussed in Section 1.3.

1.4.3 Improved Sensing Capabilities

To properly respond to changes in its environment, cognitive radios will need to be able to detect and classify the signals in its environment. If deployed in an opportunistic manner, it will be important for the cognitive radios to differentiate between primary spectrum licensees whose signals must be protected from interference and from other opportunistic signals for which less complicated measures are required. Additionally, there may be a variety of different primary signals in the same band, each of which can handle a different level of interference. For example in the UHF bands in the US which have been suggested for initial cognitive radio deployments, there are currently three primary signals that must be protected - analog TV, digital TV, and wireless microphones – with the possibility of many more in the future.

Somewhat repeating the process when spread spectrum moved from the military sphere to the commercial market, many of the needed technologies already exist, but are not publicly known. However, public researchers are now actively exploring the issue of signal detection and classification with initial promising results from combinations of algorithms that exploit cyclostationarity properties to extract signal information and neural networks to make sense of the information [Fehske_05].

Even with the best sensing capabilities, there exists the possibility of failing to find the operating primary devices due to hidden node problems. To help combat this, a variety of solutions have been proposed [Brown_05] including maintaining spectrum usage tables, network assisted detection, and placing beacons on the primary license devices. Of these approaches, network assistance (wherein cognitive radios share their observations in the network) and spectrum usage tables (updateable by primary and secondary service providers) appear to be the most promising approaches. The IEEE 802.22 standardization committee is currently considering requiring the maintenance of spectrum usage tables as a part of its standard [Cordeiro_05].

1.4.4 Software Radio Issues

As cognitive radio is just an evolution of the software radio control processes, all software radio issues will also be issues for cognitive radio. This includes improving frequency flexibility and agility, enhancing data converter technologies and careful software architecting.

Frequency flexibility and agility is critical to successful implementation of opportunistic spectrum utilization. While MEMS controlled RF devices should soon be able to provide both high performance and rapid RF reconfiguration, an intermediate solution may be available now using FETs to implement the same switches that would be used with MEMS. [Oh_04] has proposed the use of FETs to implement reconfigurable antennas and [Domalapally_04] has proposed the use of FETs to implement reconfigurable oscillators and anti-aliasing filters. Using FET-controlled RF, cheap reconfigurable RF can be achieved now with a clear upgrade path to MEMS.

To sense available spectrum and other signals in the environment, wider bandwidth ADCs will be needed. Advances in data converter technologies appear to have accelerated recently [Le_05] so this may not be a significant limitation. Likewise improved processors will greatly aid the development of the intelligent routines needed the advanced topology routines, learning, and environmental models. This too appears to be on a promising path with multiple core solutions being adopted by Intel and taken to the logical extreme by PicoChip whose picoArray contains hundreds of ARM processors.

However, one of the more important unsolved issues facing cognitive radio is operational validation. As is the case for software radio, validating software is an NP-complete problem, i.e., for complete certainty in operation, every possible combination of inputs must be tried. For a cognitive radio expected to operate in many different environments with millions of possible adaptations, this could be a very lengthy process. While a number of different entities have recognized the importance of developing techniques for validating cognitive radio designs and implementations, e.g., testing for acceptable interference is the topic of IEEE 1900.2 and software module qualification is the subject of IEEE 1900.3, no generalizable techniques have yet been developed.

1.4.5 Interactive Cognitive Radios

While even minimally cognitive radios hold great promise, there is some concern that cognitive radios may negatively impact network performance. While how a cognitive radio can negatively impact network performance may not be immediately apparent from cognition cycle shown in Figure 1.1, a more realistic diagram of the processes of a cognitive radio in a network is shown in Figure 1.18 where cognitive radios react to both "dumb" and cognitive radios. Specifically, many cognitive radios will be reacting to an outside world whose state is jointly determined by the adaptations of several cognitive radios, making any network of two or more cognitive radios an interactive decision process.

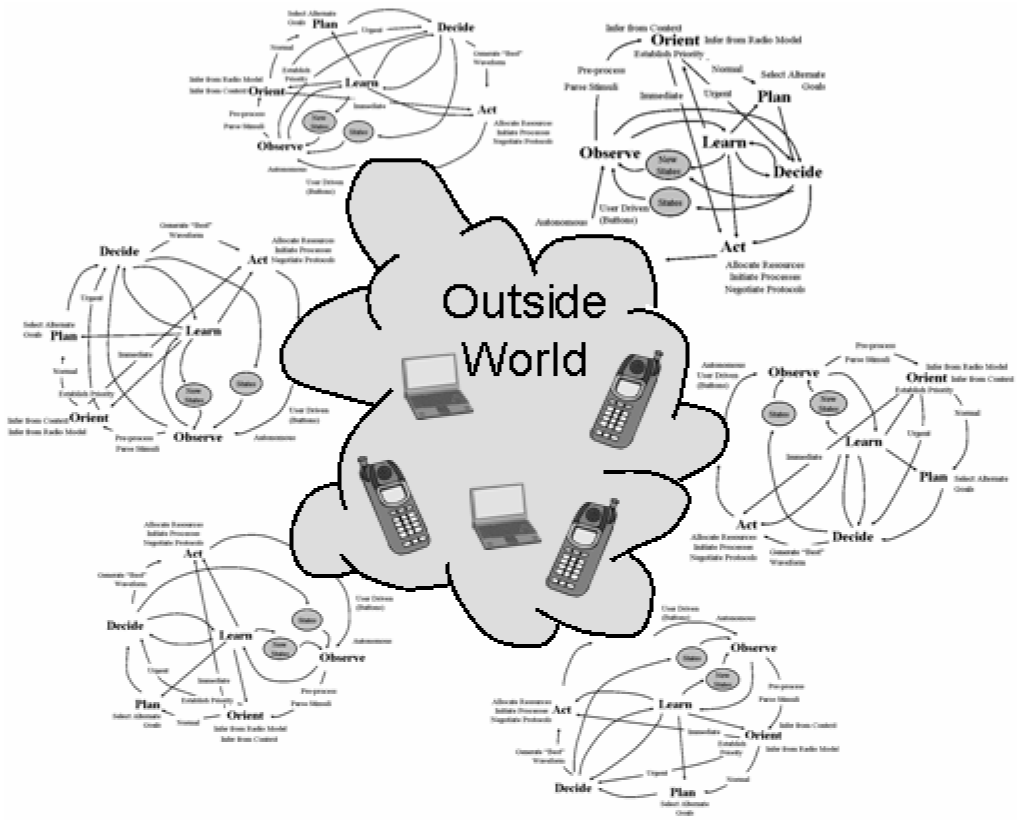

Figure 1.18: The interactive cognitive radio model. Reproduced from Figure 15-1 in [Neel_06a].

While we intuitively understand the reaction of a cognitive radio to a collection of "dumb" radios, the interaction of a collection or cognitive radios is less clear as each cognitive radio waveform adaptation changes the state of the outside world for all the other radios. The actions of a collection of cognitive radios would then appear as a recursive interactive decision process as adaptation spawns adaptation after adaptation, perhaps infinitely as implied by Figure 1.19. Such an infinite process of adaptations makes performance guarantees difficult to make and networks nearly impossible to plan in a traditional sense. Further, while some authors have proposed having the receiver dynamically determine the adaptations of the transmitter; it seems more reasonable that any adaptations will be performed at least with the knowledge of the receiver, if not actually directed by the receiver. So an infinite recursion of adaptations may imply poor utilization of spectrum as bandwidth is consumed to signal these adaptations.

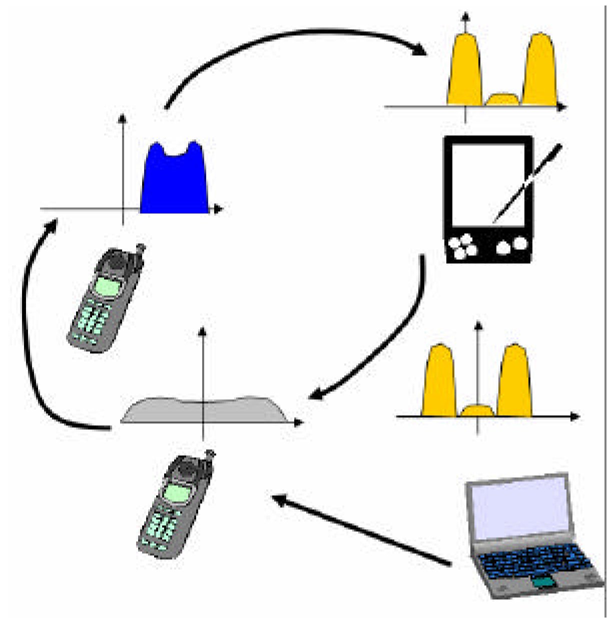

Figure 1.19: A network of adaptive radios that has fallen into an infinite adaptation recursion.

Even when these adaptations do not continue infinitely, the final state of the network might be quite undesirable. For instance, consider a single cluster DS-SS network with a centralized receiver where all nodes other than the centralized receiver are adjusting their transmitted power levels in an attempt to maximize their signal-to-interference-plus-noise ratio (SINR) as measured at the receiver. The initial state in terms of transmit power levels (blue) and SINR (green) for this network are shown in Figure 1.20. Following this implied adaptation scheme, the final state for this network is shown in Figure 1.21 where all terminals are transmitting at their maximum power levels. Clearly this is an undesirable outcome as (1) capacity is greatly diminished due to near-far problems (unless the nodes are all at the same radius from the receiver) and counter to a goal of MANET operation, (2) the resulting SINRs are unfairly distributed (the closest node will have a far superior SINR to the furthest node), and (3) battery life would be greatly shortened.

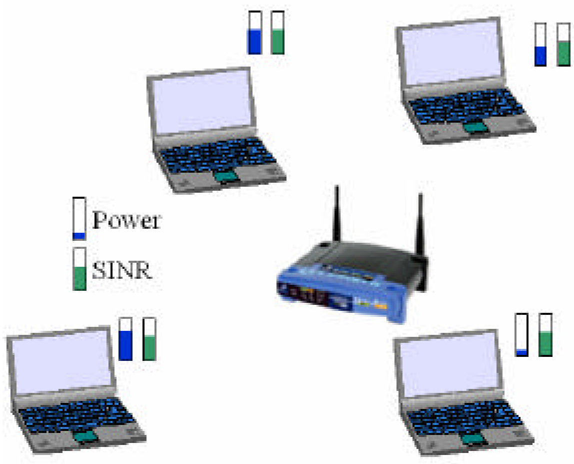

Figure 1.20: Initial network state.

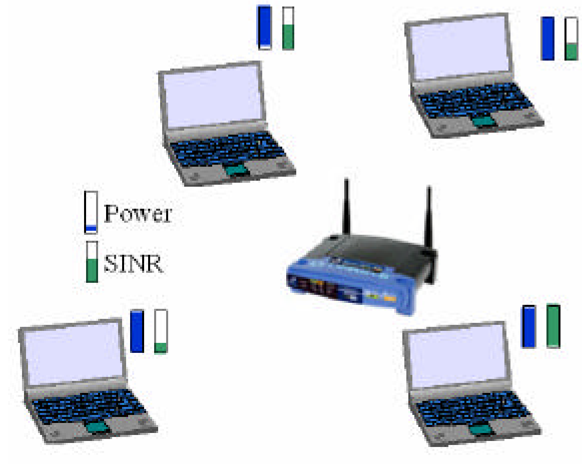

Figure 1.21: Final network state.

Abstracting the problem of interactive cognitive radios, consider a network of three radios where repeated adaptations define out paths in the action space (the combined set of all possible choices of waveforms by the three cognitive radios). Sometimes these paths terminate in a stable point; under different conditions the paths may enter an infinite loop. There may also be points in the action space which are fixed points of the decision update rule but are unstable as any small perturbation in initial conditions drive the network away from the point. Each of these concepts is illustrated in the example interaction diagram shown in Figure 1.22 where paths are shown by the arrows and fixed points are labeled as "NE" in reference to "Nash equilibrium" – a concept introduced in Chapter 4.

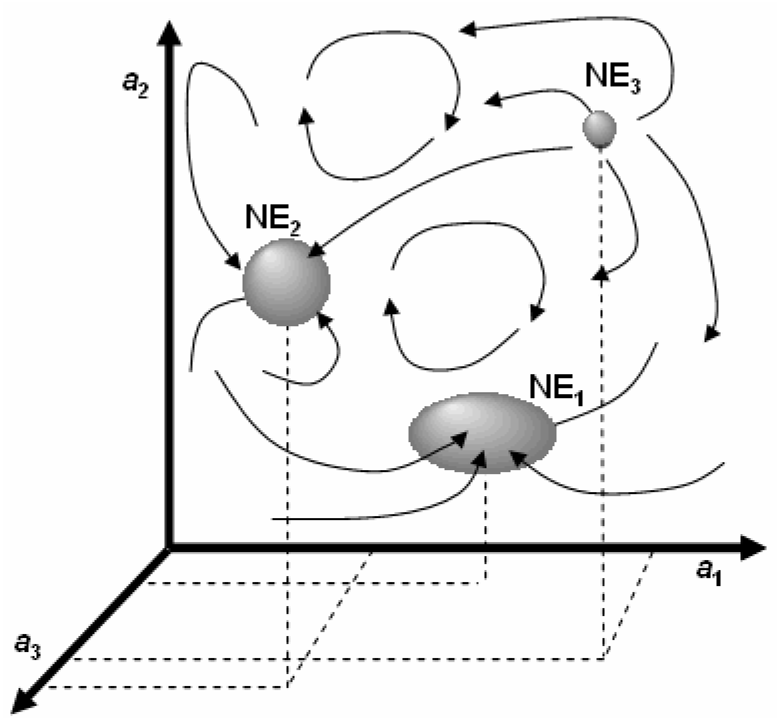

Figure 1.22: A three radio interaction diagram with three steady states (NE_1, NE_2, and NE3) and adaptation paths.

This conceptual interaction diagram illustrates the four different analysis questions that we would like to answer when considering a network of interactive cognitive radios.

- What is the expected behavior of the network?
- Does this behavior yield desirable performance?
- What conditions must be satisfied to ensure that adaptations converge to this behavior?
- Is the network stable?

To answer these questions, several researchers [Neel_06a] [MacKenzie_01] have proposed the use of game theory to analyze the interactions of autonomous adapting wireless devices.

1.5 Problem Statement and Research Contributions and Document Organization

This section refines the problem addressed by this work, describes the contributions made as part of this work

1.5.1 Problem Statement

This research addresses the issue presented in Section 1.4.5 – how can we ensure that cognitive radio algorithms will behave well in a network? Tackling this issue requires us to handle three inter-related issues.

- How do we model an interactive cognitive radio network?
- How do we analyze an interactive cognitive radio network?
- How do we design an interactive cognitive radio network?

1.5.1.1 Modeling

Modeling a cognitive radio network is a non-trivial task as cognitive radios can be implemented as either procedural or ontological radios and both implementation classes may be present in a single network. Thus to accurately model a cognitive radio network, we need models that simultaneously capture the adaptations and interactions of ontological and procedural (both deterministic and non-deterministic) radios. Further this model should be amenable to a wide variety of possible networking architectures, decision timings, waveform adaptations (possibly governed by policies), and operating environments. Of course, our models should also facilitate our analysis and design efforts.

1.5.1.2 Analysis

When analyzing a cognitive radio network, the interactions of a cognitive radio network can be viewed as creating a recursion of adaptations that modify the network state. As highlighted in Section 1.4.5, we wish to be able to analyze the recursions of cognitive radio algorithms to answer the following questions.

- Will the recursion have a fixed point (steady state) and can we identify the steady-state (or steady-states) so we can anticipate performance?
- Will that performance be desirable?
- What conditions will be necessary to ensure convergence?
- Will the steady-states be stable or will the inherent variations of the wireless medium make the system unpredictable?

While we could attempt to address these issues via simulation and experimentation, this will be a very time consuming task even for limited systems considering limited scenarios. For example, in [Ginde_03], a desktop simulation of a modeled GPRS network that incorporated power and rate adaptations required days to fully simulate all possible combinations of powers and rates for a system with just seven subscriber units in a fixed position. Expanding this simulation to account for more units, different positions or even mobility would have required months of simulation time.

Instead, we would prefer to be able answer our questions in just minutes by mathematically analyzing the structure and characteristics of the interactions of cognitive radios algorithms. As such, the goal of this research is to present a methodology suitable for quickly analyzing many cognitive radio networks with interactive and recursive decision processes with a particular focus on the kinds of cognitive radio algorithms that are deployed today – transmit power control and adaptive interference avoidance.

Further, rather than effectively reinventing the wheel for each new network and algorithm, if our analysis can follow a model-based approach analytical effort can be more efficiently spent on establishing results for generalizable models and model identification criteria.

1.5.1.3 Design

If we are only able to model and analyze the interactions of cognitive radio networks, that would be a useful result in and of itself. However, the design of cognitive radio networks would remain a hit-or-miss affair as we would not know how a network would perform until we analyze it.

We would prefer to be able to leverage our insights from modeling and analyzing cognitive radio algorithms to formulate algorithm design rules that result in behavior that converges to stable desirable steady-states.

1.5.2 Research Contributions

This research presents an application-independent model of cognitive radio interactions which we can refine to application specific models dependent on the algorithms being studied. Addressing the analysis issues required the development of new models, new analysis results for contraction mappings, new applications of analysis techniques to cognitive radio algorithms, and the development of design frameworks. Techniques for analyzing procedural radios for determining steady-states, desirability, and stability are introduced based on dynamical systems, contraction mappings, and Markov chains.

A game theoretic approach is proposed for the analysis of ontological radios and this is shown to be applicable to procedural radios as well. This research also refines two attractive game models – potential games and supermodular games – so they become suitable candidates for analysis of cognitive radio algorithms which required significant work developing the theoretical convergence and stability implications of these models as well as novel identification criteria. These approaches are applied to dynamic frequency selection (DFS) and transmit power control (TPC) algorithms – the two algorithms most commonly discussed for use in cognitive radios. An additional study of a self-configuring sensor network is also presented.

These modeling and analysis results are leveraged to develop new algorithm design rules for cognitive radio networks. These rules are shown to yield cognitive radio networks that are low complexity, scalable, convergent to optimal performance, and suitable for implementation in either procedural or ontological radios.

1.5.3 Document Organization

The remainder of this document is organized as follows.

Chapter 2: Presents a model developed as part of this work suitable for modeling cognitive radio interactions. This model can be applied to all known cognitive radio algorithms and implementations and is amenable to a wide variety of analysis techniques. This model is used in all subsequent chapters.

Chapter 3: Discusses techniques for analyzing procedural cognitive radios. The chapter addresses dynamical systems theory, contraction mappings, and Markov chain theory. Techniques for establishing steady-states, optimality, convergence, and stability are presented.

Chapter 4: Describes how game theory can be used to model procedural and ontological cognitive radios. Normal form games and repeated games are covered. General game theoretic techniques for establishing steady-states, optimality, convergence, and stability are presented including the concepts of Nash equilibria (NE), Pareto optimality, and the Finite Improvement Property.

Chapter 5: Presents the theory of potential games which are particularly well suited as a design framework for ontological radios. The chapter shows how potential games simplify NE identification, introduces techniques for guaranteeing optimal performance, and exhibit broad convergence and stability conditions. Several novel game theoretic results are introduced.

Chapter 6: Leveraging potential game theory, proposes a novel framework for designing cognitive radio algorithms – the Interference Reducing Networks (IRN) framework. This framework is shown to result in behavior that minimizes sum network interference and is shown to be implementable with either procedural or ontological cognitive radios.

Chapter 7: Focuses on a particular realization of the IRN framework for Dynamic Frequency Selection (DFS) intended for implementation on procedural or ontological cognitive radios. This algorithm is a low complexity highly scalable algorithm that only requires local observations, yet reduces the interference of all cognitive radios in the network.

Chapter 8: As part of a process of introducing examples of a key game theory concept (weak FIP, this chapter presents the theory of supermodular games which are particularly well suited as a design framework for procedural radios. A commonly encountered class of ad-hoc power control algorithms is shown to be a supermodular game, and a sensor network algorithm is proposed and shown to have weak FIP.

Chapter 9: Based on the modeling and analysis covered in the preceding chapters, this chapter draws conclusions on the design and implementation of cognitive radio networks and summarizes the results of this dissertation.

Original research contributions are made in every chapter in this dissertation. Sometimes an entire chapter is an original contribution. Other chapters present theory which needed refining for application to cognitive radios. For chapters where original and previous related work are interspersed, original definitions and theorems are marked by an asterisk. Table 1.2 lists major original contributions to the modeling, analysis, and design of cognitive radio interactions made as part of this work. Papers and awards resulting from this research are listed in Chapter 9.

Table 1.2: Major Novel Contributions Made as Part of this Work

Chapter	Research Contributions
Chapter 1	Definition of procedural and ontological cognitive radios. Definition of waveform
Chapter 2	General model of cognitive radio interactions
Chapter 3	Application of dynamical systems to the analysis of procedural radios Stability of standard interference function (SIF) Application of SIF to ad-hoc networks
Chapter 4	Application of game theory to cognitive radios General game model of cognitive radio networks Novel random better response algorithm with broader convergence conditions Convergence analysis for basic game theoretic properties under different decision timings Ergodic Markov chain model of noisy cognitive radio networks Necessary condition for convergence of myopic rational cognitive radios
Chapter 5	Application of potential games to wireless network design Multilateral Symmetric Interference Games Identification of ordinal potential games via better response

	transformations Convergence of round-robin/random better response algorithms for potential games with infinite action spaces Convergence of asynchronous better response algorithms for finite action spaces Stability of potential games for discrete time adaptations
Chapter 6	Interference Reducing Network (IRN) design framework Global altruism algorithm Local altruism algorithm Bilateral Symmetric Interference identification condition General algorithm for implementing an IRN in an isolated cluster Close proximity algorithm Impact of legacy devices
Chapter 7	Novel Dynamic Frequency Selection algorithm for ad-hoc networks and its performance under non-ideal circumstances
Chapter 8	Condition for uniqueness and stability of supermodular games A convergence proof of typical ad-hoc TPC algorithms Novel sensor network formation algorithm

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Chapter 2: Modeling and Problem Formalization

"If we can really understand the problem, the answer will come out of it, because the answer is not separate from the problem." - Jiddu Krishnamurti

Before proceeding to develop solution techniques for cognitive radios interactions, we must first define what we wish to solve. This chapter presents a general model of the interactions of cognitive radios applicable to both procedural and ontological radios and refines the aspects of their behavior that we wish to be able to analyze.

2.1 A General Model of Cognitive Radio Interactions

Consider the interactive cognitive radio problem previously illustrated in Figure 1.18 and repeated in Figure 2.1. In this interaction problem, each radio reacts to observations of the outside world by choosing some adaptation (or waveform) that the radio believes will help bring it closer to its goal, whatever that goal may be. At any given point in time, the observation a cognitive radio makes will be a function of the *passive operating environment* of the network (the channel conditions and interference environment that would be observed if no cognitive radios were operating in the environment) and the decision processes of the cognitive radios – decision processes that may be implemented via procedures or via a reasoning engine. Regardless of the implementation of the decision process, by definition, the cognitive radios are guided in their adaptations by some goal.

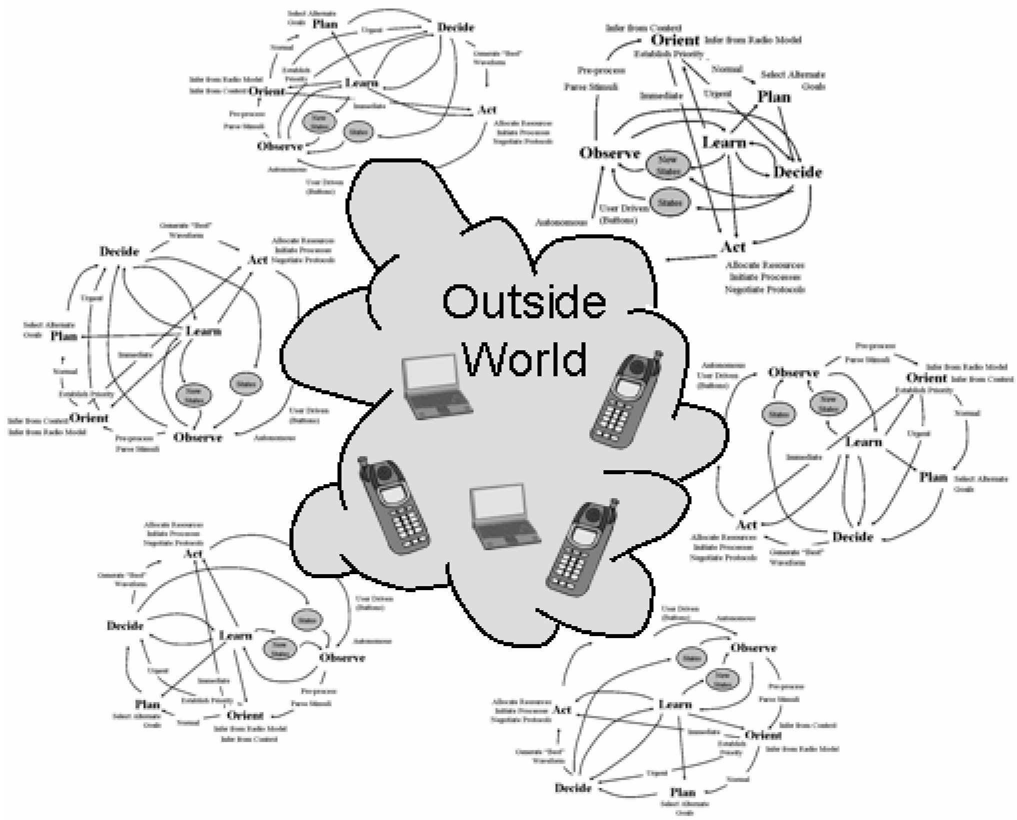

Figure 2.1: The interactive cognitive radio problem. [Neel06]

While application specific variables and models are introduced as needed later in this document, the following presents a collection of symbols and conventions that captures the general features of cognitive radio interaction and can be fashioned into a usable model of cognitive radio interactions.

- N – The (finite) set of cognitive radios in the network where n is the number of elements in N, $|N|$.
- i, j – Particular devices in N.
- A_j – The set of actions available to radio j. While these sets are quite limited for many radios, they include all available adaptations to the radio. As the adaptations can include a number of independent types of adaptations, e.g., power levels, modulations, channel and source coding schemes, encryption algorithms, MAC algorithms, center frequencies, bandwidths, and routing algorithms, A_j will generally be a multidimensional set.

To simplify matters, we assume we are analyzing adaptations only over a short time interval so the A_j will not be a function of time, i.e., the radios are not learning new actions while they are adapting. This is consistent with the earlier discussion that cognitive radios' learning processes are expected to be performed during sleep or prayer processes [Mitola_00]. However, if we consider time scales that spanned these sleep or prayer processes, A_j could be expected to grow as radio j learns new waveforms.

- A – The *action space*, i.e., set of all possible combinations of actions by the radios in the network. Throughout, we assume that A is formed by the Cartesian product of each radio's action sets, i.e., $A = A_1 \times A_2 \times \cdots \times A_n$. For some algorithms, it is convenient to think of A as a vector space with orthogonal bases A_1 through A_n.

- a – A particular combination of actions where each radio in N has implemented a particular action (waveform), i.e., a point in A or an *action vector*. Radio j's contribution to a is written as a_j, and the choice of actions by all cognitive radios other than j is written as a_{-j}.

- O – The set of all possible *observed outcomes* of the outside world as determined by the choice of actions available to each cognitive radio and the passive operating environment.

- o_j – An observation made by or supplied to radio j. For instance, an SINR measurement.

- o – A vector of observed outcomes where all radios have observed an outcome. For instance, o may represent a vector of SINR measurements with each measurement associated with a particular cognitive radio. Frequently, we refer to this as an *outcome.*

- d_j – The *decision rule* which describes how radio j updates its decisions based on observations.

Strictly, d_j is a function that relates actions to outcomes, i.e., $d_j : o_j \rightarrow A_j$. However, while the observed outcome may only be statistically related to the action vector, we will assume for the purposes of analysis that the relationship between actions and observed outcomes is known and treat each decision rule as a function that relates action vectors to actions, i.e., $d_j : A \rightarrow A_j$.

For procedural cognitive radios, the decision rule may be explicitly given; for ontological cognitive radios, we may have to make broad generalizations such as the implemented decision rule selects a *locally optimal* action or the radio behaves *selfishly*. A more formal treatment of decision rules for ontological radios is presented in Chapter 4. However, throughout this report, we assume that each radio's decision rule is guided by its goal or *utility function*.

- $u_j(a)$ – The *utility function* which describes how much value radio j assigns to action vector a. Throughout this report we assume these values or utilities are described using real numbers, i.e., $u_j : A \rightarrow \mathbb{R}$. In general, the utility function expresses some goal that the radio is working towards whether explicitly (ontological cognitive radio) or implicitly (procedural cognitive radio).

Again, a practical implementation of a cognitive radio's goal would associate numbers with the radio's observed outcomes, o_j, and not the action vector as other radios' actions will not generally be directly observable. However, for purposes of analysis we assume that the analyst knows the relation between actions and observed outcomes so that the analyst can express the utility function in terms of the action vector. Therefore, for analysis purposes, these utility functions capture the actions of the cognitive radios and the passive operating environment.

Although elided in the introduction to this section, the exact times at which radios make their decisions can significantly influence the behavior of a network. In military circles, there is much effort placed on getting inside the enemy's decision loop because of the potential advantages gained by the quicker decision maker. Or in a more mundane circumstance, anyone who has met someone head on in a hallway and proceeded to repeatedly block each others' attempts to pass knows the effects that there is a significant difference between synchronous and asynchronous decision timings. To capture the cognitive radio equivalent of these conditions, our model requires addition of the following symbols and conventions.

- T_j – The times at which radio j can update its decision (a radio may have a time allocated for updating, but choose not to update its decision). Unless stated otherwise, we assume that each T_j is infinite, i.e., $T_j = \{t_j^0, t_j^1, \ldots, t_j^m, \ldots\}$. As we are ultimately modeling interactive software processes, we always assume that T_j is a discrete set.
- T – The set of all times where decision updates can occur, i.e., $T = T_1 \cup T_2 \cup \cdots \cup T_n$, where $t \in T$ denotes a particular updating time. For convenience, we treat t^k as the k^{th} element of T arranged chronologically.

Further, when appropriate, we also use the notation d^t to denote the *network decision rule* at time t where in general d^t captures the adaptations of the subset of radios that update their decisions at time t, i.e., $d^t = \times_{k \in M} d_k$, $M \subset N$. While it is also possible that a radio bases its decisions on past observations and predictions about the future state of the network, this text assumes that d_j^t is only a function of cognitive radio j's most recent observation.

Additionally, we make use of the following terms in describing the timing of the decision update process: *synchronous decision processes*, *round-robin decision processes*, *random decision processes*, and *asynchronous decision processes*.

Definition 2.1: *Synchronous decision process*
If $\forall t \in T$, $d^t = \times_{k \in N} d_k$, then we say that the network has a *synchronous decision process* and write $a^{t_{k+1}} = d^{t_k}\left(a^{t_k}\right)$.

Definition 2.2: *Round-robin decision process*
If $t_1^m < t_2^m < t_3^m < \cdots < t_n^m < t_1^{m+1}$, then we say that the network is updating its decisions in *round-robin* order.

Definition 2.3: *Random decision process*

If $\forall t \in T \;\; d^t = \underset{k \in N}{rand}\{d_k\}$, then we say that the network is updating its decisions in *random* order

Definition 2.4: *Asynchronous set decision process*

If $\forall t \in T \;\; d^t = \underset{k \in 2^N}{rand}\{d_k\}$, then we say that the network is updating its decisions in *random* order

For asynchronously updating networks, there may be some points in time where $t_i^m = t_j^k$ (the m^{th} update of radio i occurs at the same time as the k^{th} update of radio j) is satisfied for two or more radios. As we will see in subsequent sections, these different decision update timings – synchronous, round robin, random, and asynchronous – can have a significant impact on the analysis of our cognitive radio network.

Systems with synchronous timings are most frequently encountered in centralized systems and thus will be rarely encountered in an interactive cognitive radio decision process as an interactive decision process implies some degree of distributed decision timings. A round-robin scheme can occur in centralized systems with distributed decision making with scheduling (as might occur in a hybrid ARQ scheme). Without a synchronizing agent and assuming an arbitrary fineness in the time scale, every distributed cognitive radio algorithm will be a randomly decision process. However, because real-world observations necessitate processing data collected over non-infinitesimal intervals and because of signal propagation delays, a system with random timings will behave more like an asynchronous system.

Summarizing this discussion, the basic model of cognitive radio interaction consists of a collection of a collection of cognitive radios, N, an action space A, a utility function for each cognitive radio $j \in N$ which is a function of the actions of each radio and the passive operating environment, a decision rule for each cognitive radio, and a set of times at which these decisions occur. This can be compactly represented as the 5-tuple shown in (2.1).

$$\left\langle N, A, \{u_j\}, \{d_j\}, T \right\rangle \tag{2.1}$$

The following chapters illustrate how this model can be applied to networks of procedural and ontological radios. For procedural radios, we place increased modeling emphasis on the decision rules; for ontological radios, we place increased emphasis on the radios' goals. If we ignore the mapping from actions to outcomes, our model is implementation independent, though not particularly useful for analysis. With the mapping from actions to outcomes in place, our model is implementation specific – useful for analysis, though difficult to generalize.

Example 2.1: Example: Modeling a Cognitive Radio Algorithm

Consider two cognitive radios, {1,2}, with actions (waveforms) $\{w_{1_a}, w_{1_b}\}$ and $\{w_{2_a}, w_{2_b}\}$, respectively, that are communicating with a common receiver which reports to each cognitive radio that radio's *signal-to-interference ratio* (SIR). In this scenario, there are four different possible elements in *A*, which form the set $\{(w_{1_a}, w_{2_a}), (w_{1_a}, w_{2_b}), (w_{1_b}, w_{2_a}), (w_{1_b}, w_{2_b})\}$. However, there are an infinite number of possible observations due to the infinite number of passive operating environments.

In this case, the passive operating environment is defined by the gains from each cognitive radio to the common receiver, g_1 and g_2. We'll consider the interference that one waveform induces on the other to be given by the absolute value of the correlation of their signal space representations, $\left| r\left(w_j, w_{-j} \right) \right|$ where w_j is the waveform chosen by radio $j \in \{1,2\}$ and w_{-j} is the waveform chosen by the other radio. In such a system, the observed outcome for each radio *j* is given by the SIR equation given in (2.2) where $j \in \{1,2\}$, g_j is the observed SIR for radio *j*, g_j is the link budget gain of radio *j* to the common receiver, and g_{-j} is the gain of the other radio to the common receiver.

$$o_j = g_j = \frac{g_j}{g_{-j}\left|r\left(w_j, w_{-j}\right)\right|} \tag{2.2}$$

A reasonable goal or a utility function for a cognitive radio operating in this system would be to maximize (2.2) so that the greater the SIR the radio achieves, the higher the value the radio assigns to the outcome. Note that this goal incorporates both the relevant information from the passive operating environment (in this case, the link gains), the potential actions that could be taken by the radios, and the interactive nature of those actions.

Particularly as each radio only has two waveforms to choose from, it seems reasonable to assume that whether procedurally or ontologically each radio implements a locally optimal decision rule or more formally as given in (2.3).

$$d_j(a) = \underset{w_j \in \{w_{ja}, w_{jb}\}}{\operatorname{argmax}} \frac{g_j}{g_{-j}\left|r\left(w_j, w_{-j}\right)\right|} \tag{2.3}$$

Finally, by controlling when observations are returned to the cognitive radios, the common receiver could conceivably enforce any decision timing scheme. However, this example will assume that adaptations occur in a round robin fashion with one adaptation permitted each half second, e.g.., $T_1 = n$ sec and $T_2 = n + 0.5$ sec where $n \in \mathbb{N}$. Based on this discussion, these various modeling parameters can be compactly summarized as shown in Table 2.1.

Table 2.1: Parameters for Example Model

General Model Symbols	Modeled System Parameters
N (cognitive radio set)	$\{1,2\}$
A (action space)	$\left\{\left(w_{1_a},w_{2_a}\right),\left(w_{1_a},w_{2_b}\right),\left(w_{1_b},w_{2_a}\right),\left(w_{1_b},w_{2_b}\right)\right\}$
$\{u_j\}$ (utility functions)	$u_j\left(a\right)=\dfrac{g_j}{g_{-j}\left\lvert r\left(w_j,w_{-j}\right)\right\rvert}$
$\{d_j\}$ (decision rules)	$d_j\left(a\right)=\underset{w_j\in\left\{w_{ja},w_{jb}\right\}}{\operatorname{argmax}}\, u_j\left(a\right)$
T_j (decision timings)	$T_2 - 0.5\text{s} = T_1 = \mathbb{N}$

2.2 Analysis Objectives

By using these modeling parameters and considering another example of cognitive radio interaction, we can begin to formalize our analysis objectives. Consider a network of three radios where each radio, $j \in \{1,2,3\}$, can choose actions from a convex action set, A_j, according to its decision update rule d_j. Starting at any initial action vector, the repeated application of the decision update rules trace out *paths* in the action space.

Definition 2.5: *Path*
A path is a sequence of action vectors, $\left\{a^{t_k}\right\}$ formed by the recursion $a^{t_{k+1}} = d^{t_k}\left(a^{t_k}\right)$.

Note that even if the same network decision rule and the same passive operating environment are used, different paths result from different initial points, a^0.

Conceptually, a path may terminate in a stable point, but under different conditions a path may enter an infinite loop. There may also be points in the action space that are fixed points of the decision update rule but are unstable so that any small perturbation in initial conditions drives the network away from the point. Each of these concepts is illustrated in the example interaction diagram shown in Figure 2.2 where paths are shown by the arrows and fixed points are labeled as "NE" for reasons that will become clear in Chapter 4.

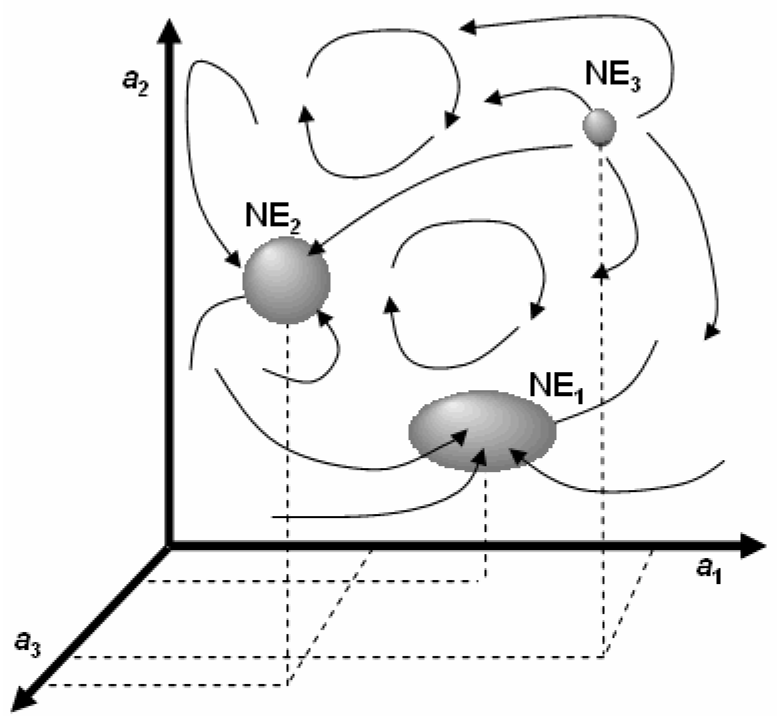

Figure 2.2: A three radio interaction diagram with three steady states (NE_1, NE_2, and NE3) and adaptation paths.

This conceptual interaction diagram illustrates the four different analysis questions we identified in Chapter 1 that we would like to answer when analyzing the interactions of a network of cognitive radios.

1. What is the expected behavior of the network?
2. Does this behavior yield desirable performance?
3. What conditions must be satisfied to ensure that adaptations converge to this behavior?
4. Is the network stable?

The following formalizes the analysis objectives underlying each of these questions and previews some of the techniques introduced later in this text.

2.2.1 Establishing Expected Behavior

As is the case for many systems, the analysis in this report assumes that expected behavior of a cognitive radio network is equivalent to its steady-state behavior. Accordingly, establishing expected behavior is concerned with addressing the following issues:

- *Existence* – Does the system have a steady state?
- *Identification* – What are the specific steady states for the system?

In general, we will consider an action vector, $a^* \in A$, to be a steady state for a network if it is a fixed point of the decision rule, a condition that is expressed more formally in Definition 2.6.

Definition 2.6: *Steady state*
An action vector a^* is a steady state for the cognitive radio network $\langle N, A, \{u_j\}, \{d_j\}, T \rangle$ if there is some $t^* \in T$ such that for all $t \geq t^*$, $d^t(a^*) = a^*$.

Subsequent chapters will introduce a number of different techniques for demonstrating that a steady state exists and for identifying the steady states of the network. These include showing that the network decision rule is a variant of a *contraction mapping*, that the network can be modeled as an *absorbing Markov chain*, and that the network obeys certain game theoretic properties.[1]

2.2.2 Desirability of Expected Behavior

Of course, determining a cognitive radio network's steady states tells us nothing about whether or not we should implement the algorithm under study. We also need to address whether or not those steady states are "good" steady states of "bad" steady states and if there are other action vectors that would be preferable from a network designer's perspective. Again, there are two specific issues that we would like to address:

- *Desirability* – How "good" are the steady states of the algorithm?
- *Optimality* – Does an optimal action vector exist and how close do the steady states come to achieving optimal performance?

There are many different ways of identifying whether or not an action vector is a "good" steady state, but we will make the assumption that the network designer has some objective function, $J : A \rightarrow \mathbb{R}$ that he/she wishes to maximize or minimize (perhaps total system goodput or spectrum utilization). Assuming we wish to maximize *J*, we'll treat

[1] Contraction mapping and absorbing Markov chain are defined in Chapter 3. The associated game theoretic techniques are defined in Chapters 4 and following.

action vector a^2 as more desirable than a^1 if $J\left(a^2\right) > J\left(a^1\right)$. To determine if an optimal action vector exists and if our steady states are indeed optimal, subsequent chapters will introduce gradient techniques and *Pareto optimality* criteria.

2.2.3 Convergence Conditions

Even if we demonstrate that a cognitive radio network has desirable steady states, it is important to identify the conditions (decision rules, passive operating environments, initial conditions) under which paths *converge*, a concept formalized in Definition 2.7.

> **<u>Definition 2.7</u>**: *Convergence*
> Given path $\left\{a^{t_k}\right\}$, we say that the path *converges* to some action vector $a^* \in A$ if for every $e > 0$, there is a $t^* \in T$ such that $t \geq t^*$ implies $\left\| a^{t^*}, a^* \right\| < e$.

In other words, path $\left\{a^{t_k}\right\}$ converges to a^* if for every arbitrarily small region around a^* that we might define, there is a time after which $\left\{a^{t_k}\right\}$ remains "trapped" in that region.

For convergence, this research addresses the following issues:

- *Rate* – Given $\left\{a^{t_k}\right\}$ and $e > 0$, what is the value of t^* such that the path converges?
- *Sensitivity* – How do changes in the value of a^0, slight variations in d^t (perhaps asynchronous instead of round-robin timings) or changes in the passive operating environment impact the paths and the network's steady states?

Frequently when assessing convergence this text considers a time-independent decision rule, *d*, coupled with varying timings for implementing decision rule. For example, this text considers time-independent decision rules corresponding to locally optimal decisions, directional improvement, and randomly selected better responses coupled with synchronous, asynchronous, random, or round robin decision timings. This approach allows us to establish the sensitivity of the decision rules to timing variations more precisely.

2.2.4 Network Stability

Implicitly, the preceding analysis objectives assume the radios have perfect knowledge of their operating environment and behave deterministically. However, wireless networks are stochastic, not deterministic. Accordingly, the cognitive radios' observations will not be the deterministic functions and instead will be estimates of their operating environment. Because these are only estimates, the radios will frequently make adaptations that appear to be mistakes to the analyst. While this research assumes the radios' estimates and errors are unbiased, there is the concern of stability as small perturbations could potentially lead to undesirable behavior. Because of this concern, this research addresses the following analytical issues with respect to a network decision rule's steady state(s):

- *Lyapunov stability* – After a small perturbation, will stay the system within a bounded region about the steady state?
- *Attractivity* – After a small perturbation, will the network converge back to the steady state?

2.3 Summary

This chapter has presented a generalized model of cognitive radio interactions and identified important analysis objectives. This model is defined by the tuple $\langle N, A, \{u_i\}, \{d_i\}, T \rangle$ where the associated symbols are summarized in Table 2.2. Subsequent chapters provide application-specific refinements of this model and introduce techniques for determining steady states, desirability of those steady states, convergence criteria, and stability.

Table 2.2: Symbol Summary

Symbol	Meaning	Symbol	Meaning
N	Set of cognitive radios	i, j	Particular cognitive radios
A_j	Adaptations for j	a_j	Adaptation chosen by j
a_{-j}	Adaptation vector excluding a_j	u_j	Goal of j
O	Set of outcomes	O_j	Outcome observed by j
d_j	Decision rule for j	T_j	Times when j adapts
T	Adaptation times $\forall j \in N$	t	An element of T

In general we will seek to design cognitive radio algorithms such that all of their steady-states maximize the design objective for the particular application, are converged to and are stable under the broadest possible conditions, and require a minimal amount of signaling overhead and device resources to realize the algorithm. While this seems to be an impossible order to fill, by leveraging the analysis insights of the subsequent chapters, Chapters 6, 7, and 8 present several algorithms that meet all of these objectives.

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Chapter 3: Tools for Analyzing the Interactions of Procedural Radios[1]

"Before thinking outside the box, one should know what's in the box. The box tends to have a lot of good ideas – that's how they came to be in the box."

In this chapter we consider the problem of analyzing the interactions procedural radios based on the model presented in Chapter 2. In general, we can study the interactions of procedural radios via a reduced model that excludes their goals, namely the tuple $\langle N, A, \{d_j\}, T \rangle$. As this chapter shows, many traditional analysis techniques from engineering can be applied to the analysis of procedural radios, including dynamical systems theory, optimization theory, parallel processing (contraction mappings), and Markov chain theory. Before cognitive radio, before SDR, these techniques were being applied to the analysis of wireless algorithms. And as we show in this chapter, they are still useful for the analysis of procedural cognitive radios. Further, when we turn to the analysis of ontological radios in subsequent chapters many of the concepts presented in this chapter resurface.

The remainder of this chapter is organized as follows. Section 3.1 considers dynamical systems and describes how the *evolution function* can be used to determine steady-states, optimality, convergence, and stability. Section 3.2 presents variants on contraction mappings, including the standard interference function and pseudo-contractions, and describes how they can be used to determine steady-states, optimality, convergence, and stability. Section 3.3 presents Markov chain theory which can be used to determine steady-states, optimality, convergence, and stability for non-deterministic procedural radios.

3.1 A Dynamical Systems Approach

Dynamical systems theory is concerned with analyzing the behavior of dynamical systems and designing mechanisms so the systems act in a desirable manner. Typical

[1] This chapter is based on a section in J. Neel. J. Reed, A. MacKenzie, "Cognitive Radio Network Performance Analysis" in **Cognitive Radio Technology**, B. Fette, ed., Elsevier, July 28 2006.

analysis goals of dynamical systems theory are similar to the ones that we set out in Chapter 2: determining the expected behavior, convergence, and stability of the system.

Formally, a dynamical system is a system whose change in state is determined by a function of the current state and time. In other words, a dynamical system is any system of the form given by (3.1) which describes the change in the state of a system as a function of the system state, a, and time, t. Implicitly the system is assumed to be at state $a(0)$ at time t=0.

$$\dot{a} = g(a,t) \tag{3.1}$$

When (3.1) is not directly dependent on t, i.e., $\dot{a} = g(a)$, the system is said to be *autonomous*. For our purposes, it makes sense to treat synchronous systems as autonomous, but for random and asynchronous systems, it is difficult to eliminate the time dependency.

The first goal of a dynamical systems analyst is to solve (3.1) to yield the *evolution function* that describes the state of the system as a function of time. This typically involves solving an ordinary differential equation – a task that we would preferably not undertake without knowing that a solution exists. Given a dynamical systems model, we can be assured that such a solution exists by the Picard-Lindelöf theorem [Walker_80].

Theorem 3.1: Picard-Lindelöf Theorem
Given an open set $D \subset A \times T$ and g as in (3.1), if g is continuous on D and *locally Lipschitz* with respect to a for every $a \in D$, then there is a unique solution, d^t, to the dynamical system for every $a(0)$ while d^t remains in D.

Note that Theorem 3.1 requires that g is not only continuous, but also locally Lipschitz – a term we define in Definition 3.1. Note that any function that is Lipschitz continuous is also continuous.

Definition 3.1: *Lipschitz continuity*

A function, $d^t : A \times T \to A, A \subset \mathbb{R}^n$ [2] is said to be *Lipschitz continuous*[3] at (a,t) if there exists a $K < \infty$ such that $\left\| d^t(a^1,t) - d^t(a^2,t) \right\| \leq K \left\| a^1 - a^2 \right\|$ for all $a^1, a^2 \in A$; d^t would be *locally Lipschitz continuous* if this condition were only satisfied for some open set $D \subset A \times T$. Similarly the function d is *Lipschitz continuous* if it is Lipschitz continuous for all $(a,t) \in A \times T$.

In general, the solution to (3.1) will take the form of the decision update rule, d^t, which we assumed existed as part of our model. So this section primarily serves the purpose of connecting our model to the model traditionally assumed in dynamical systems. However, Theorem 3.1 foreshadows the importance of fixed point theorems to the steady-states of procedural radio networks.

3.1.1 Fixed Points and Solutions to Cognitive Radio Networks

A solution for the evolution function d^t may imply a system that is changing states over time, perhaps bounded within a certain region or wandering over the entire action space. For some systems, continual adaptations may not be an issue and may even be desirable. However, continual adaptations for a cognitive radio network implies that significant bandwidth is being consumed to support the signaling overhead required to support these adaptations.

For a cognitive radio network, we would prefer that the network settle down to a particular steady state and only adapt as the environment changes. Identifying these steady-states also allows a cognitive radio designer to predict network performance. In the context of our state equation, such a steady state is a *fixed point* of d^t.

Definition 3.2: Fixed Point

A point $a^* \in A$ is said to be a *fixed point* of $d^t : A \to A$ if $a^* = d^t(a^*) \forall t \geq t^*$.

[2] d^t is a function that maps from the Cartesian Product of the action space with the set of all update times to the action space, where the action space is a subset of all real *n*-tuples; that is, given an initial action state, it forms the action space over all time.

[3] A function, d^t, is Lipschitz continuous if there exists a finite real K, such that for all action states a^1 and a^2 in the action space, the Euclidean distance between $d^t(a^1)$ and $d^t(a^2)$ is less than K times the distance between a^1 and a^2.

For one dimensional sets, it is convenient to envision a fixed point of a function as a point where the function intersects the line $x = f(x)$. Figure 3.1 illustrates a function, *f*(*x*), that has three fixed points.

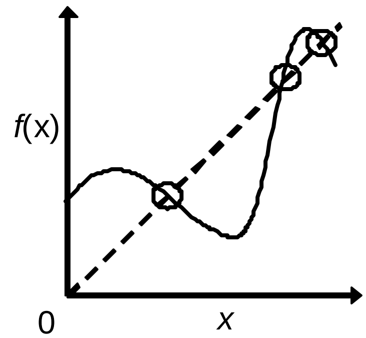

Figure 3.1: A function with three fixed points (circled). For functions on a one dimensional sets, the points at which the function intersect the line *f*(*x*) = *x* (dashed) are fixed points.

Solving for fixed points can be tedious as it may involve a search over the entire action space (an impossibility for an infinite action space, and a considerable undertaking for most realistic finite action spaces), so we would like to know if a fixed point exists before we begin our search. Fortunately, this can be readily established by the Leray-Schauder-Tychonoff fixed point theorem given by Proposition 1.3 in Chapter 3 of [Bertsekas_97].[4]

Theorem 3.2: Leray-Schauder-Tychonoff Fixed Point Theorem
If $A \subset \mathbb{R}^n$ is nonempty, convex, and compact (see Appendix B), and if $d: A \to A$ is a continuous function, then there exists some $a^* \in A$ such that $a^* = d^t(a^*)$.[5]

However, there are several limitations to this theorem. First, the theorem is inappropriate for finite action sets – a likely condition – as while finite sets are compact, they are not convex. Second, *d* may not be a continuous function. Third, actually solving for a fixed point under such general conditions can be much harder, though under these conditions we simultaneous solution of (3.2) is appropriate for identifying steady-states.

[4] The game theorist may recognize this as a variant of Brouwer's fixed point theorem.
[5] Leray-Schauder-Tychonoff actually considers continuous *mappings* instead of continuous function, but a continuous function is a continuous mapping.

$$a_i^* = d_i\left(a^*\right) \forall i \in N \tag{3.2}$$

3.1.2 Establishing Optimality

Perhaps the easiest way to establish that a solution to a cognitive radio network is optimal is to show that it max(min)imizes some objective function $J : A \to \mathbb{R}$. For networks with a finite action space we can perform an exhaustive search and evaluate *J* at each point in *A*.

However, this approach is impractical for infinite action spaces. But when *J* is differentiable and *A* is a compact interval of $\mathbb{R}^n$, we can reduce the search space by noting that if a particular action vector, a^*, is optimal, then a^* must either be a boundary point or a point where $\nabla J\left(a^*\right) = 0$ where $\nabla J\left(a\right) = \frac{\partial J\left(a\right)}{\partial a_1}\hat{a}_1 + \frac{\partial J\left(a\right)}{\partial a_2}\hat{a}_2 + \cdots + \frac{\partial J\left(a\right)}{\partial a_n}\hat{a}_n$ [6] where each $\hat{a}_j$ is a dimension of *A*. So in effect, this condition says that for a^* to optimize *J*, there must be no direction that can be followed from a^* that increases *J*. If *J* is *pseudo-concave,* we can change this to a sufficient condition, i.e., if there exists some point such that $\nabla J\left(a^*\right) = 0$, then it is optimal.[7] [Zangwill_69]

> **Definition 3.3**: *Pseudo-concavity*
> A function $J : A \to \mathbb{R}$ is said to be *pseudo-concave* if $\nabla J(a'') \cdot (a' - a'') \leq 0 \Rightarrow J\left(a'\right) \leq J\left(a''\right)$ for all points $a', a'' \in A$.

More familiarly, a function that is *concave* is also pseudo-concave.

> **Definition 3.4**: *Concavity*
> A function, $J : A \to \mathbb{R}$, is concave on the set *A* if for all $a_1, a_2 \in A$, $J\left(l a_1 + (1-l) a_2\right) \geq l J\left(a_1\right) + (1-l) J\left(a_2\right)$ for all $l \in [0,1]$.

---

[6] The gradient of the cost function *J*, is in general a vector valued function that when evaluated at a particular point, *a''* indicates the magnitude and direction of greatest increase for *J* at *a''*. When *J* is a function of a single dimension, then the gradient of *J* is equivalent to the slope of *J*.

[7] This is a variant on the Karush-Kuhn-Tucker theorem.

Equivalently, a function is concave if it is impossible to join two points in the function with a line that contains points above the function.

Figure 3.2 shows an example of a function that is pseudo-concave, but not concave. This function can be verified to not be concave by considering a line joining the points (0, 0) and (1,1) (shown as a dashed line); except for the endpoints, all of the points in this line lie above the function.

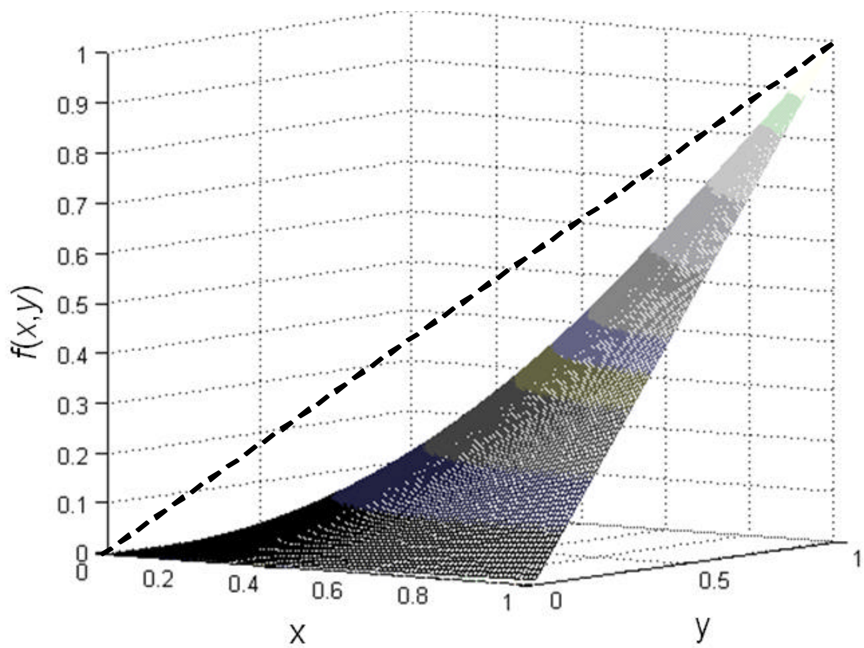

Figure 3.2: $f(x, y) = xy$, $x, y > 0$ – A function that is pseudo-concave, but not concave.

3.1.3 Convergence and Stability

When discussing convergence and stability of a decision rules fixed point, it is convenient to make use of two forms of stability: *Lyapunov stability* and *attractivity.*

> **Definition 3.5**: *Lyapunov stability*
> We say that an action vector, a^*, is *Lyapunov stable* if for every $e > 0$ there is a $d > 0$ such that for all $t \geq t^0$, $\left\| a\left(t^0\right), a^* \right\| < d \Rightarrow \left\| a\left(t\right), a^* \right\| < e$ [8].

[8] Equivalently, the action vector a^* is said to be Lyapunov stable if for every arbitrarily sized ε>0, it is possible to identify a δ>0 such that after a pertubation to any point $a(t^0)$, all subsequent action vectors are no more than a Euclidean distance of ε away from a^*.

While no particular relation between d and e can be inferred from this definition, an engineer may be more comfortable thinking of Lyapunov stability as akin to Bounded-Input-Bounded-Output stability wherein after a bounded "stimulus" of d is added to a system operating at a^*, the system remains within a bounded distance e of a^*.

Definition 3.6: *Attractvity*
The action vector a^* is said to be *attractive* over the region $S \subset A$, $S = \{a \in A \mid \|a, a^*\| < M\}$, if given any $a(t_0) \in S$, the sequence $\{a(t)\}$ converges to a^* for $t \geq t_0$.

Tying both concepts together is the *asymptotic stability.*

Definition 3.7: *Asymptotic Stability*
The action vector a^* is said to be *asymptotically stable* if it is both Lyapunov stable and attractive.

Note that Lyapunov stability does not imply attractivity nor does attractivity imply Lyapunov stability. For instance, the fixed point (0,0) in Figure 3.3 is Lyapunov stable, but not attractive; meanwhile the fixed point (0,0) in Figure 3.4 is attractive, but not Lyapunov stable. However, the intuition that both stability and attractivity are frequently found together is borne out by Lyapunov's Direct Method.

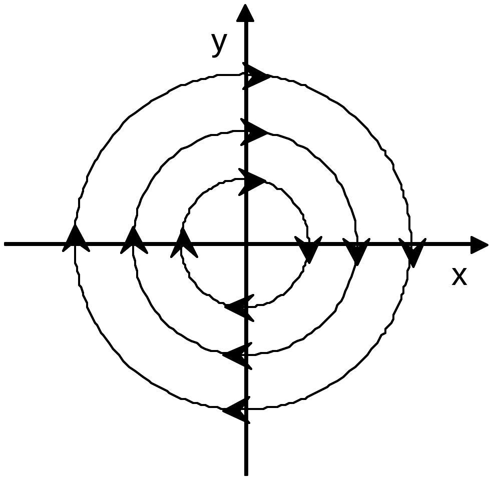

Figure 3.3: Paths (formed by recursive application of d^t with direction indicated by arrows) for a system that is Lyapunov stable but not attractive.

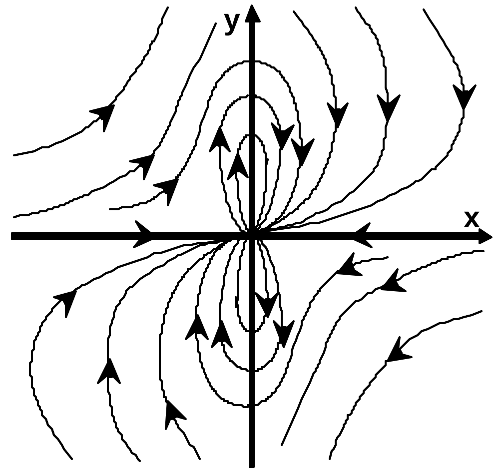

Figure 3.4: Paths for a fixed point that is attractive but not Lyapunov stable.

Instead of attempting to directly apply the definitions of Lyapunov stability and asymptotic stability, we can use Lyapunov's direct method. The discrete time version of Lyapunov's direct method is given in Theorem 3.4 of [Medio_01] as follows.

Theorem 3.3: *Lyapunov's Direct Method for Discrete Time Systems*

Given a recursion $a\left(t^{k+1}\right)=d^{t}\left(a\left(t^{k}\right)\right)$ *with fixed point* a^{*}, *we know that* a^{*} *is Lyapunov stable if there exists a continuous function (known as a* Lyapunov function*) that maps a neighborhood of* a^{*} *to the real numbers, i.e.,* $L:N(a^{*})\rightarrow\mathbb{R}$, *such that the following three conditions are satisfied:*

1) $L\left(a^{*}\right)=0$
2) $L(a)>0\ \forall a\in N\left(a^{*}\right)\setminus a^{*}$
3) $\Delta L\left(a(t)\right)\equiv L\left[d^{t}\left(a(t)\right)\right]-L\left(a(t)\right)\leq 0\ \ \forall a\in N\left(a^{*}\right)\setminus a^{*}$

Further, if conditions 1-3 hold and

a) $N\left(a^{*}\right)=A$, *then* a^{*} *is globally Lyapunov stable;*
b) $\Delta L\left(a(t)\right)<0\ \forall a\in N\left(a^{*}\right)\setminus a^{*}$, *then* a^{*} *is asymptotically stable;*
c) $N\left(a^{*}\right)=A$ *and* $\Delta L\left(a(t)\right)<0\ \forall a\in N\left(a^{*}\right)\setminus a^{*}$, *then* a^{*} *is globally asymptotically stable.*

Lyapunov's direct method says, in effect, that if we can find a function that strictly decreases along all paths created by the adaptations of a cognitive radio network, then that cognitive radio network is asymptotically stable.

It is also interesting to note that the existence of a Lyapunov function can be used to establish the existence and identify the network's steady-states, namely all points where $L\left(a^*\right)=0$. Further, Lyapunov's direct method can be readily applied to both synchronous and asynchronous cognitive radio networks – the only requirement being each adaptation must decrease the value of the Lyapunov function. Of course, there are cognitive radio algorithms with many steady-states which are so closely spaced that it is impossible to identify any combination of neighborhood and Lyapunov function that meets these definitions. Such a scenario is considered in Chapter 7.

While this section does not present a particular example analysis of an evolution equation for fixed points, convergence, optimality, or Lyapunov stability, these concepts will be repeatedly applied throughout the remainder of this document.

3.2 Contraction Mappings and the General Convergence Theorem

In the preceding discussion, we assumed a closed form expression for the next network state as a function of current network state. Now suppose that after one recursion of the network update rule we are unable to precisely predict the next network state. However, we are able to bound the network state within a particular set of states $A(t^1)$. Then suppose that armed with the knowledge that the network starts in $A(t^1)$, we could say that after the second iteration, the network state would have to be within another set $A(t^2)$, which is a subset of $A(t^1)$. Extending this concept, suppose that given any set of network states, $A(t^k)$, we know that the decision update rule always results in a network state in the set, $A(t^{k+1})$, which is a subset of $A(t^k)$.

In effect, this process is saying that as the recursion continues finer and finer approximations on the operating point of the network are possible, perhaps resulting in a prediction of a specific steady-state for the network. Such a sequence of finer approximations might look as shown in Figure 3.5 where the recursion of subsets, $A(t^k)$, converges to a single point. This iterative restriction on a recursion's possible points forms the basis of numerous valuable algorithms and is a characteristic of special class of algorithms known as *contraction mappings*.

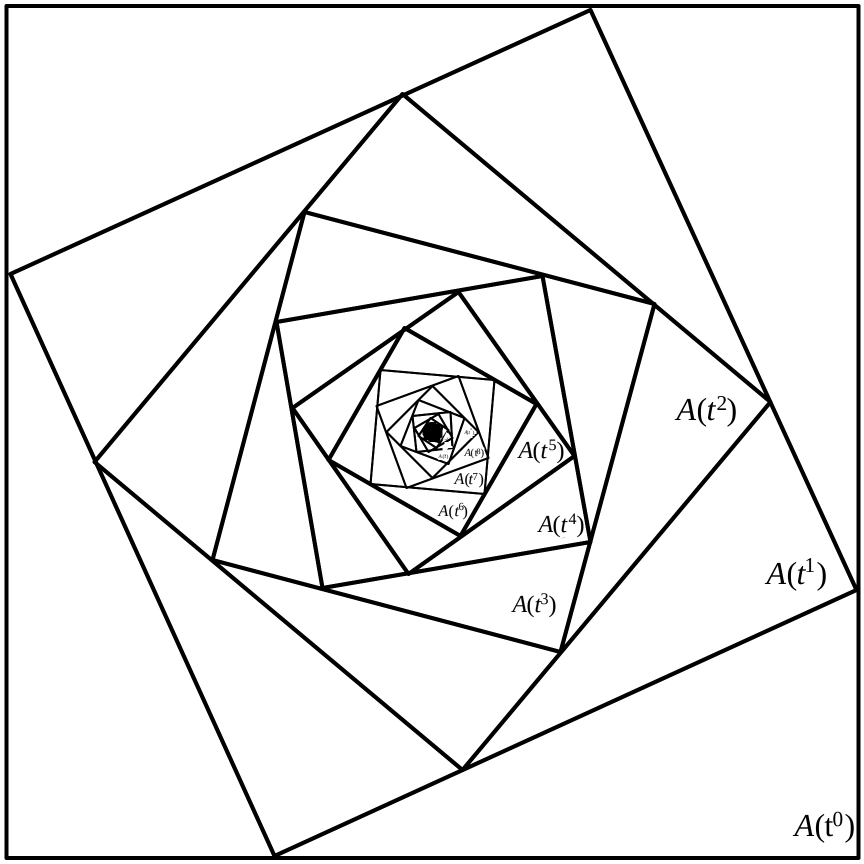

Figure 3.5: A sequence of contracting sets, $\cdots \subset A(t^2) \subset A(t^1) \subset A(t^0)$.

3.2.1 Contraction Mappings

Definition 3.8: *Contraction mapping*

Given a recursion $a(t^{k+1}) = d(a(t^k))$, d is said to be a *contraction mapping* with modulus α if there is an $a \in [0,1)$ such that $\|d(a), d(b)\| \le a \|a, b\| \;\; \forall b, a \in A$.

While applying this definition to a decision rule can be difficult, we can show that an arbitrary recursion, *d*, is a contraction mapping if the following two conditions are satisfied: [Blackwell_65]

Theorem 3.4: *Blackwell's conditions*
Given recursion, $a\left(t^{k+1}\right)=d\left(a\left(t^{k}\right)\right)$, *d* forms a contraction mapping if *d* satisfies monotonicity and discounting.
1) *Monotonicity* – Given bounded functions $g_1, g_2 : A \to \mathbb{R}$ where $g_1(a) \le g_2(a)$ $\forall a \in A$, *d* must satisfy $d\left(g_1(a)\right) \le d\left(g_2(a)\right)\ \forall a \in A$.
2) *Discounting* – There exists a $b \in (0,1)$ such that $d\left(g_1(a)+c\right) = d\left(g_1(a)\right) + bc$ for all bounded $g_1 : A \to \mathbb{R}$, $c \ge 0$, $a \in A$.

3.2.2 Analysis Insights

Knowing that our decision rule constitutes a contraction mapping immediately provides us with several valuable insights. From Banach's contraction mapping theorem[9] given in [Sundaram_99], we know that *d* has a unique fixed point to which the recursion *f* converges from any starting point. After *k* iterations, a bound on the distance of the current state from the fixed point is given by (3.3).

$$\left\| a\left(t^{k}\right), a^{*} \right\| \le \frac{a^{k}}{1-a} \left\| a\left(t^{1}\right), a\left(t^{0}\right) \right\| \qquad (3.3)$$

(3.3) is also useful for bounding the error in estimating *d* 's fixed point by recursively evaluating *d*. Additionally, a Lyapunov function for any contraction mapping with fixed point a^* is given by (3.4). Thus every contraction mapping, *d*, has a unique stable fixed point to which *d* converges at a predictable rate.

$$L(a) = \left\| a, a^{*} \right\| \qquad (3.4)$$

3.2.3 Pseudo-Contractions

A pseudo-contraction eliminates the contraction mapping's requirement that all points move closer to each other after each iteration but still requires that after each iteration, all points move closer to a unique fixed point.

Definition 3.9: *Pseudo-contraction*
Given mapping $d : A \to A$ with fixed point, a^*, we say *d* is a *pseudo-contraction* if there is an $a \in [0,1)$ such that $\left\| d(a), d\left(a^{*}\right) \right\| \le a \left\| a, a^{*} \right\|\ \forall a \in A$.

9 Banach's fixed point theorem is simply that every contraction mapping has a unique fixed point.

By definition, d has a unique fixed point, a^*, to which d converges at a rate given by (3.5). Note that evaluation of (3.5) requires knowledge of the fixed point, so unlike (3.3), it is not appropriate for bounding the error on an estimate of the system's fixed point while iterating to solve for the fixed point. Also note that (3.4) serves as a Lyapunov function for a pseudo-contraction and that a^* is globally asymptotically stable.

$$\left\| a\left(t^k\right), a^* \right\| \le \alpha^k \left\| a\left(0\right), a^* \right\| \tag{3.5}$$

3.2.4 General Convergence Theorem

For most contraction mappings, it is assumed that the updating process occurs synchronously (recall the discussion of decision timings in Chapter 2). We can relax this assumption by introducing the general convergence theorem presented in Proposition 2.1 of Chapter 6 in [Bertsekas_97].

Theorem 3.5: *General Convergence Theorem*

Suppose we know that $\cdots \subset A\left(t^{k+1}\right) \subset A\left(t^k\right) \subset \cdots \subset A\left(t^0\right)$ *where* $A(t^k)$ *represents the possible states of the network after k iterations and* $A\left(t^0\right)$ *represents all possible initial states for the network. Then if the following two conditions hold, then f also converges asynchronously.*

1)*Synchronous Convergence Condition*

(a) $d\left(a\right) \in A\left(t^{k+1}\right)\ \forall k, a \in A\left(t^k\right)$

(b) If $\{a(t^k)\}$ *is a sequence such that* $a\left(t^k\right) \in A\left(t^k\right)$ *for every k, then every limit point of* $\{a(t^k)\}$ *is a fixed point of d.*

2) Box Condition

For every k, there exist sets $A_j\left(t^k\right) \subset A_j$ *such that* $A\left(t^k\right) = A_1\left(t^k\right) \times \cdots A_n\left(t^k\right)$.

For our purposes, the general convergence theorem states that under an assumption that each radio's action sets are independent (thereby implying the action space satisfies the box condition), any contraction or pseudo-contraction mapping that converges synchronously also converges asynchronously. However, we can also apply the general convergence theorem to algorithms that are not obviously contraction mappings as seen in the following extended example.

3.2.4.1 Standard Interference Function Model

Many traditional analyses consider specific decision rules that model specific applications. The following discusses such an analysis that is also an example of a non-obvious contraction mapping. [Yates_95] considers a power control algorithm operating on the uplink of a cellular system with uniform frequency reuse.[10] For this algorithm, there is a set of N mobiles where each mobile, j, attempts to achieve a target received SINR, $\hat{g}_j$. The development of this algorithms assumes that each mobile is capable of observing its received SINR (perhaps via feedback from a base station) which is generally given by (3.6) where g_{kj} can be the link budget gain from mobile k to the base station of j, p_k is the transmit power of mobile k, and N_j is the noise power at the base station that is receiving mobile *j's* signal.

$$g_j = \frac{g_{jj} p_j}{\sum_{k \in N} g_{kj} p_j + N_j} \tag{3.6}$$

Based on observations of (3.6), the mobiles compute a scenario dependent *interference function*, $I_j(\mathbf{p})$, which is formed as the ratio of the target SINR, $\hat{g}_j$, and the effective SINR, g_j, i.e., as shown in (3.7)

$$I_j(\mathbf{p}) = \hat{g}_j / g_j \tag{3.7}$$

where $\mathbf{p}$ is the vector of transmit powers, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, drawn from the power vector space $\mathbf{P}$.

Generalizing beyond this ratio formalization, Yates defines any interference function to be *standard* if it satisfies the conditions given in Definition 3.10 where we write $\mathbf{p}^1 \geq \mathbf{p}^2$ if $p_j^1 \geq p_j^2 \; \forall j \in N$ and $I(\mathbf{p})$ is the synchronous evaluation of all $I_j(\mathbf{p})$.

[10] It is not stated in [Yates_95] that uniform frequency reuse is an assumption; rather this is the result of a conversation with Yates at DySPAN 05 about [Yates_95].

Definition 3.10: Standard Interference Function
An interference function, $I : \mathbf{P} \rightarrow \mathbb{R}^n$, is said to be standard if it satisfies the following three conditions:

1. *Positivity* - $I(\mathbf{p}) > 0$
2. *Monotonicity* – If $\mathbf{p}^1 \geq \mathbf{p}^2$ then $I(\mathbf{p}^1) \geq I(\mathbf{p}^2)$
3. *Scalability* – For all $a > 1$, $a I(\mathbf{p}) > I(a\mathbf{p})$

Assuming the existence of a standard interference function, Yates defines a synchronous updating process of the form $\mathbf{p}(t^{k+1}) = d(\mathbf{p}(t^k))$ where $d(\mathbf{p}) = d_1(\mathbf{p}) \times \cdots \times d_n(\mathbf{p})$ and f_j is given by (3.8).

$$d_j(\mathbf{p}(t^k)) = p_j(t^k) I_j(\mathbf{p}(t^k)) \tag{3.8}$$

[Yates_95] then considers the situation where the target SINR vector, $\hat{g} = (\hat{g}_1, \ldots, \hat{g}_n)$, is *feasible*.

Definition 3.11: Feasible SINR Vector
A target SINR vector, $\hat{g}$, is said to be *feasible* if there exists a $\mathbf{p} \in \mathbf{P}$ such that $g_j \geq \hat{g}_j \; \forall j \in N$.

When the, [Yates_95] is able to show that an algorithm updating the power vector according to (3.8) has the following properties:

1. A fixed point exists, i.e., there is some $\mathbf{p}^*$ such that $\mathbf{p}^* = d(\mathbf{p}^*)$
2. This fixed point is unique
3. Starting from any initial power vector, d converges to $\mathbf{p}^*$.

While [Yates_95] shows these results in an ad-hoc manner, [Berggren_01] shows that this updating process constitutes a pseudo-contraction which could be used to establish these same results by applying the results of Section 3.2.3. Further we would also know that d is stable. The fact that d constitutes a pseudo-contraction implies that $\cdots \subset \mathbf{P}(t^{k+1}) \subset \mathbf{P}(t^k) \subset \cdots \subset \mathbf{P}(t^0)$ where $\mathbf{P}(t^k)$ is the power vector space for iteration k. Coupled with the just established synchronous convergence of f and implicit satisfaction of the box condition, this means that d has satisfied the conditions for the general convergence theorem. Thus it is known that d converges both synchronously and asynchronously. These results are proven in a more rigorous fashion using different techniques in Chapter 9.

3.2.4.2 Further Insights from the Standard Interference Function

Assuming the SINR feasibility criterion is satisfied, [Yates_95] also shows that the following target SINR arrangements of base stations and mobiles have standard interference functions and thus converge synchronously and asynchronously to a unique power vector when the decision update rule is given as in (3.8).

- Fixed assignment – each mobile is assigned to a particular base station;
- Minimum power assignment – each mobile is assigned to the base station in the network where the mobile's SINR is maximized
- Macro diversity – all base stations in the network combine the signals of the mobiles;
- Limited diversity – a subset of the base stations combine the signals of the mobiles; and
- Multiple connection reception – the target SINR must be maintained at a number of base stations.

Feasible SINR

Previously we defined a target SINR vector, $\hat{g}$, as being feasible if there exists a $\mathbf{p} \in \mathbf{P}$ such that $g_j \geq \hat{g}_j \; \forall j \in N$. Rather than performing an exhaustive search over **P**, [Zander_01] presents the following approach for determining if $\hat{g}$ is feasible.

Consider a network with link gain matrix **G** formed as $\mathbf{G} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix}$ where g_{jk} is the link gain as used in (3.8). Now form the normalized link matrix H as $h_{ij} = g\dfrac{g_{ji}}{g_{ii}}, i \neq j$ with $h_{ii} = 0$. Then p. 155 of [Zander_01] tells that the uniform target

SINR vector $\hat{g}_u = (\hat{g}, \hat{g}, \ldots, \hat{g})$ is achievable if the spectral radius (largest eigenvalue)[11] of **H** is less than or equal to one. When the spectral radius is exactly 1, then $\hat{g}$ is achievable only when there is no noise in the system. Interestingly, [Berggren_01] states that for the pseudo-contraction formed by the standard interference function, $a = r(\mathbf{H})$ which allows us to estimate the convergence rate as well.

A similar expression can be found for the nonuniform target SINR scenario where $\hat{g} = (\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_n)$ as follows where the link matrix, ***H'***, is formed as $h'_{ij} = \frac{\hat{g}_i}{\hat{g}_{\max}} \frac{g_{ji}}{g_{ii}}, i \neq j$ where $\hat{g}_{\max} = \max_{i \in N} \{\hat{g}_i\}$.

Assuming the target SINRs are feasible, then the power vector corresponding to the unique fixed point specific can be found by solving (3.9)

$$\mathbf{Zp} = \vec{g} \tag{3.9}$$

where $\mathbf{Z} = \begin{bmatrix} g_{11} & -\hat{g}_1 g_{21} & \cdots & -\hat{g}_1 g_{n1} \\ -\hat{g}_2 g_{1n} & g_{22} & \cdots & -\hat{g}_2 g_{n2} \\ \vdots & & \ddots & \vdots \\ -\hat{g}_n g_{1n} & -\hat{g}_n g_{2n} & \cdots & g_{nn} \end{bmatrix}$ and $\vec{g} = [\hat{g}_1 N_1 \quad \hat{g}_2 N_2 \quad \cdots \quad \hat{g}_n N_n]^T$.

3.3 Markov Models

Perhaps because of uncertainty in the order of adaptation (as would be the case for a randomly or asynchronously timed process) or because of uncertainties in the decision rules (either from noise or a non-deterministic procedural radio), it may be impossible to derive a closed-form expression for an evolution equation or to even to bound the adaptations into sequential subsets. Instead, suppose we can model the changes of the cognitive radio network from one state to another as a sequence of probabilistic events conditioned on past states that the system may have passed through. When the probability

[11] Due to the work of Hilbert, spectral theory refers to a set of theories relating to matrices, eigenvalues and eigenvectors. In spectral theory, the set of eigenvalues for a matrix is said to be its spectrum.

distribution for the next state in time, $a(t^{k+1})$, is conditioned solely on the most recent state as shown in (3.10), the random sequence of states, $\{a(t)\}$ is said to be a *Markov chain*. A model of a system whose states form a Markov chain is said to be a *Markov model*. Throughout the remainder of this section we use these two terms interchangeably.

$$P\left(a\left(t^{k+1}\right)=a^k \mid a(0),\ldots,a(t)\right)=P\left(a\left(t^{k+1}\right)=a^k \mid a\left(t^k\right)\right) \quad (3.10)$$

Formalizing our model, let us assume that our state space is finite. This is not a requirement for a Markov chain, but the assumption is useful for the subsequent discussion. Further, let us assume that if the network is in state $a^m \in A$ at time t^k, then at time t^{k+1}, the network transitions to state $a^n \in A$ with probability p_{mn} where $p_{mn} \geq 0 \; \forall a^m, a^n \in A$ and $\sum_{j \leq |A|} p_{mj} = 1$. Of course, it is also permitted that the system remains in state a^m which it does with probability p_{mm}. To simplify notation, we make use of a *transition matrix*, which we represent with symbol $\mathbf{P}$. The transition matrix is formed by assigning p_{mn} to the entry corresponding to the m^{th} row and n^{th} column.

3.3.1 Markov Model Analysis Insights

From $\mathbf{P}$ we can then form $\mathbf{P}^2$ as the matrix product $\mathbf{PP}$. Now entry p_{mn}^2 in the m^{th} row and n^{th} column of $\mathbf{P}^2$ represents the probability that system is in state a^n two iterations after being in state a^m. Similarly, if we consider the matrix $\mathbf{P}^k$ formed as $\mathbf{P}^k = \mathbf{PP}^{k-1}$ (an example of a Chapman-Kolmogorov equation for a Markov chain [Stewart_94]), then entry p_{mn}^k in the m^{th} row and n^{th} column of $\mathbf{P}^k$ represents the probability that system is in state a^n k iterations after being in state a^m.

A similar relationship can be found when the initial state is specified by a random probability distribution arranged as a column vector p where $p_m \in [0,1]$ and $\sum_{m=1}^{|A|} p_m = 1$ where p_m represents the probability of starting in state a^m. For such a situation, the state probability distribution after k iterations is given by $p^{T}\mathbf{P}^{k}$ where the superscripted T denotes the transpose operation.

3.3.2 Ergodic Markov Chains

Tying back into our analysis objectives of steady-states and convergence, we are particularly interested in determining the *stationary distributions* and *limiting distribution* of a Markov chain that models a cognitive radio network.

Definition 3.12: *Stationary Distribution*
A probability distribution such that p^{*} such that $p^{*T}\mathbf{P} = p^{*T}$ is said to be a stationary distribution for the Markov chain defined by **P**.

Note that solving for a stationary distribution is equivalent to solving the eigenvector equation given in (3.11) where λ=1.

$$p^{*T}\mathbf{P} = \lambda p^{*T} \tag{3.11}$$

Definition 3.13: *Limiting Distribution*
Given initial distribution p^{0} and transition matrix **P**, the *limiting distribution* is the distribution that results from evaluating $\lim_{k\to\infty} p^{0\,T}\mathbf{P}^{k}$.

While not generally a steady state as we considered in the previous discussion, showing that a Markov chain has a unique distribution that is both stationary and limiting would permit us to characterize the behavior of the network. Specifically, given the unique stationary limiting distribution p^{*}, we could predict that at a particular instance in time and after a sufficient number of iterations, the network would be in state a^{m} with probability π_m. Thus it is desirable to be able to identify when such a unique stationary limiting distribution exists as is done in the ergodicity theorem given in [Syski_92].

Theorem 3.6: *Ergodicity Theorem*[12]
If a Markov chain is *ergodic*, then there exists a unique limiting and stationary distribution for all initial distributions p^{0}.

[12] This is also called the "Fundamental Theorem of Markov Chains."

This theorem is in reality just a restatement of the definition of an ergodic Markov chain. However the theorem emphasizes a valuable insight – an ergodic Markov chain converges to the same limiting distribution regardless of the initial distribution. Thus when we can show that a cognitive radio network can be modeled as an ergodic Markov chain we gain the following insights:

- The network has a unique "steady-state" distribution p^*
- This distribution can be found by solving the eigenvalue problem $p^{*T}\mathbf{P} = \lambda p^{*T}$ where $\lambda=1$.
- From all initial distributions, the network converges to p^*.

[Stewart_94] states that a Markov chain is ergodic if it is a Markov chain is ergodic if it is a) *irreducible*, b) *positive recurrent*, and c) *aperiodic*.

Definition 3.14: *Irreducibility*
A Markov chain is *irreducible* if $\forall a^m, a^n \in A$, there exist sequences of state transitions with nonzero probability that lead to every state.

Definition 3.15: *Positive Recurrence*
A Markov chain is *positive recurrent* if $\forall a^m \in A$, the expected number of iterations to return to state a^m is less than ∞.

Definition 3.16: *Aperiodicity*
A Markov chain is *aperiodic* if $\forall a^m \in A$, there there is no integer, $n > 1$, such that once the system leaves the state, it can only return to the state in multiples of n iterations.

Note that a network with round-robin timing will not satisfy aperiodicity as an adaptation away from state a^m on radio *i's* turn can only return to a^m on one of radio *i*'s turns which by definition only occurs every n iterations. However, if we treat an entire round as an iteration, then the aperiodicity can be satisfied by a network with round-robin timing. In general, we prefer not to have to apply definitions to the identification process as it tends to be quite time-consuming. Fortunately theorem 4.1.2 in [Kemeny_60] provides a readily applied identification criterion.

Theorem 3.7: *Ergodocity Criteria*
A finite Markov chains with transition matrix **P** is ergodic if and only if there is some k such that $\mathbf{P}^k$ has no zero entries.

Thus by identifying this simple condition, we know that a unique identifiable stationary limiting distribution exists.

Example 3.1: Markov Model of Cognitive Radio Adaptations

Consider a network consisting of two cognitive radios where each radio can choose between two actions. This network would have four possible states which we could label $\{a^1, a^2, a^3, a^4\}$. Suppose that from experimental observation, we observe the probability transition matrix shown in (3.12) and illustrated in Figure 3.6 where each state is represented as a vertex (circle) and each transition is represented as a weighted and directed edge labeled with its associated transition probability.

$$\mathbf{P} = \begin{array}{c} \\ a^1 \\ a^2 \\ a^3 \\ a^4 \end{array} \begin{array}{c} \begin{array}{cccc} a^1 & a^2 & a^3 & a^4 \end{array} \\ \begin{bmatrix} 0.1 & 0.3 & 0.1 & 0.5 \\ 0.4 & 0.0 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 & 0.2 \\ 0.1 & 0.4 & 0.3 & 0.2 \end{bmatrix} \end{array} \quad (3.12)$$

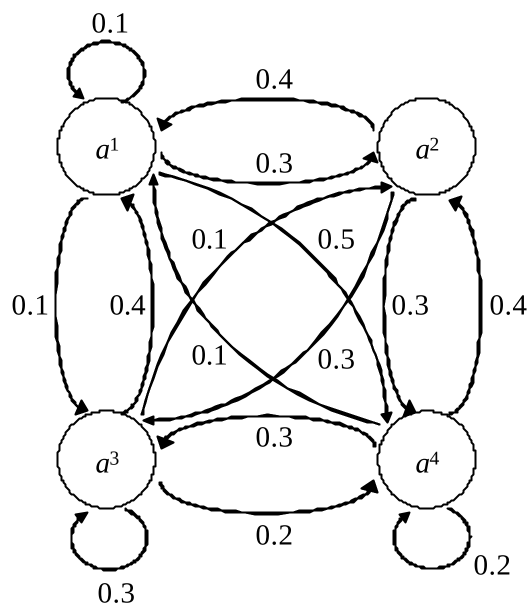

Figure 3.6 Digraph Representation of (3.12)

As specified by (3.12), **P** gives the probability of transitioning from state a^2 to state a^3 as 0.3. After calculating $\mathbf{P}^2$ as shown below, we can immediately determine the probability of the system operating in state a^4 after two iterations after starting in a^3 ($p_{34}^2 = 0.33$).

$$\mathbf{P}^2 = \begin{array}{c} \\ a^1 \\ a^2 \\ a^3 \\ a^4 \end{array} \begin{array}{c} \begin{array}{cccc} a^1 & a^2 & a^3 & a^4 \end{array} \\ \begin{bmatrix} 0.22 & 0.24 & 0.28 & 0.26 \\ 0.19 & 0.27 & 0.22 & 0.32 \\ 0.22 & 0.23 & 0.22 & 0.33 \\ 0.31 & 0.14 & 0.28 & 0.27 \end{bmatrix} \end{array}$$

Similarly, given an initial distribution of states $\mathrm{p} = [0.1 \quad 0.2 \quad 0.3 \quad 0.4]^T$, after two iterations, the probability of being in each state is given by $\mathrm{p}\,\mathbf{P}^2 = [0.25 \quad 0.203 \quad 0.25 \quad 0.297]^T$. Because all elements in $\mathbf{P}^2$ are positive, there exists a stationary distribution p^* which we can find by solving the eigenvector equation $\mathrm{p}^{*T}\,\mathbf{P} = \mathrm{p}^{*T}$ to yield $\mathrm{p}^{*T} = [0.2382 \quad 0.2352 \quad 0.2272 \quad 0.2938]$.

3.3.3 Absorbing Markov Chains

For cognitive radio networks that we can model as ergodic chains we can readily find the unique limiting distribution. However, this "steady-state" is somewhat unsatisfying as the network will not remain at a single state and all states will have nonzero probability of being occupied and thus the "steady-state" of an ergodic Markov chain does not conform to our expectations from Chapter 2. However, this is not a problem for absorbing Markov chains.

A state a^k in a Markov chain is said to be an *absorbing state* if there are no paths that leave a^k. This is defined more formally in Definition 3.17.

> **Definition 3.17**: *Absorbing State*
> Given a Markov chain with transition matrix **P**, a state a^k is said to be an absorbing state if $p_{kk} = 1$.

> **Definition 3.18**: *Absorbing Markov chain*
> A Markov chain is said to be an *absorbing Markov chain* if
> a) it has at least one absorbing state and
> b) from every state in the Markov chain there exists a sequence of state transitions with nonzero probability that leads to an absorbing state. These nonabsorbing states in are called *transient states*.

For example, (3.13) gives a transition matrix for an absorbing Markov chain where a^4 (note that p_{44}=1) is the absorbing state and a^1, a^2, and a^3 are the transient states where all transient states have a nonzero probability of transitioning directly to a^4. As we will see in later examples, the existence of a direct transition to an absorbing state is not a requirement of an absorbing Markov chain nor must a transition matrix have only a single absorbing state.

$$\mathbf{P} = \begin{array}{c} \\ a^1 \\ a^2 \\ a^3 \\ a^4 \end{array} \begin{array}{c} \begin{array}{cccc} a^1 & a^2 & a^3 & a^4 \end{array} \\ \begin{bmatrix} 0.1 & 0.3 & 0.1 & 0.5 \\ 0.4 & 0.0 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{3.13}$$

3.3.3.1 Absorbing Markov Chains Analysis Insights

Within the context of our analysis objectives, an absorbing state is a fixed point or steady state that once reached, the system never leaves. Similarly, valuable convergence insights can be gained when the system can be modeled as an absorbing Markov chain. However, establishing these convergence results requires the introduction of some additional matrices based on our transition matrix **P**.

First let us write our Markov chain transition matrix in *canonical form* which is given by the modified transition matrix, $\mathbf{P}'$ shown in (3.14) where $\mathbf{I}^{ab}$ is the identity matrix corresponding to the state transitions between the absorbing states of the chain, **Q** represents the state transitions between the nonabsorbing states of the chain, **0** is a rectangular matrix filled with all zeros representing the probability of transition from absorbing states to nonabsorbing states, and **R** represents the rectangular matrix of state transition probabilities from nonabsorbing states to absorbing states. At this point we have not performed any operations on **P**, merely relabeled the states in a way which we will find convenient.

(*canonical form*)
$$\mathbf{P}' = \left[\begin{array}{c|c} \mathbf{Q} & \mathbf{R} \\ \hline \mathbf{0} & \mathbf{I}^{ab} \end{array}\right] \tag{3.14}$$

Given $\mathbf{P}'$, Markov theory provides us with information on convergence and the expected frequency that the system visits a transitory state. First, we know that recursive evaluation of $\mathbf{P}'^{k}$ yields $\lim_{k\to\infty}\mathbf{Q}^k \to \mathbf{0}$. Recall that we earlier said that the entry p_{mn} in $\mathbf{P}^k$ represented the probability of the system initially occupying state p^k_{mn} in the m^{th} row and n^{th} column of $\mathbf{P}^k$ represents the probability that system is in state a^n k iterations after being in state a^m. Thus $\lim_{k\to\infty}\mathbf{Q}^k \to \mathbf{0}$ implies that that the probability of the system not being "absorbed", i.e., not terminating in one of the absorbing states of the chain, goes to zero.

Given an absorbing chain with a modified transition matrix as in (3.14), the *fundamental matrix* is given by (3.15).

(*fundamental matrix*) $$\mathbf{N} = \left(\mathbf{I} - \mathbf{Q}\right)^{-1} \quad (3.15)$$

Solving for the fundamental matrix **N** permits a number of valuable analytic insights. First, theorem 3.2.4 in [Kemeny_60] states that the entry n_{km} gives the expected number of times that the system will pass through state a_m given that the system starts in state a_k. Second, theorem 3.3.5 in [Kemeny_60] states that if we evaluate $\mathbf{t} = \mathbf{N}\mathbf{1}$ where **1** is a column vector of all ones, then t_k gives the expected number of iterations before the state is absorbed when the system starts in state a_k. Finally, theorem 3.3.7 in [Kemeny_60] states that if we evaluate (3.16)

$$\mathbf{B} = \mathbf{N}\mathbf{R} \quad (3.16)$$

where **R** is as given in (3.14), then entry b_{km} in **B** specifies the probability the system ends up in absorbing state a_m if the system starts in state a_k.

Thus, once we show that a Markov model for a network of cognitive radios with transition matrix **P** is an absorbing Markov chain, the following insights are readily gained:

- Steady-states for the system can be identified by finding those states a^m for which $p_{mm} = 1$.
- Convergence to one of these steady-states is assured, and the expected distribution of states can be found by solving for **B.**
- Given an initial state, a^k, convergence rate information is given by solving for **t**.

Example 3.2 describes a procedural cognitive radio DFS algorithm that can be modeled and analyzed using Markov models. It is interesting to note that with the additional stipulation that when adapting channels are chosen at random, the algorithm described in Example 3.2 can be readily scaled to any network of n radios with $c \geq n$ channels and still remain an absorbing Markov chain. However for $n > c$, the network is no longer an absorbing Markov chain and instead becomes an ergodic Markov chain. One approach to overcoming this limitation is to adjust the decision rule so that no radio switches to a channel which would be predicted to receive the same amount of interference. In such a case, the network can again be modeled as an absorbing Markov chain. Chapter 7 will consider additional techniques for ensuring desirable behavior for a DFS algorithm where $n > c$.

Example 3.2: DFS as an Absorbing Markov chain

Consider two cognitive radios implementing dynamic frequency selection over the two channel set $F=\{f_1, f_2\}$. Assume that these two radios are seeking to minimize the interference their signal and that both are implementing the simple decision rule that if an interfering signal is detected, then the radio switches to the other frequency.

Using the model from Chapter 2, this system can be modeled as $N = \{1,2\}$, $A = \{(f_1,f_1), (f_1,f_2), (f_2,f_1), (f_2,f_2)\}$, $u_j(a) = \begin{cases} 1 & f_j \neq f_{-j} \\ -1 & f_j = f_{-j} \end{cases}$, $d_j(f_j, f_{-j}) = \begin{cases} f_j & u_j(a) = 1 \\ f \in F \setminus f_j & u_j(a) = -1 \end{cases}$, and T is asynchronous where due to a random timer for each $t \in T$ each radio gets a chance of 0.5.

This model can then be converted into a Markov model with the transition matrix shown in (3.17) and illustrated in Figure 3.7.

$$\mathbf{P} = \begin{array}{c} \\ (f_1,f_1) \\ (f_1,f_2) \\ (f_2,f_1) \\ (f_2,f_2) \end{array} \begin{array}{c} \begin{array}{cccc} (f_1,f_1) & (f_1,f_2) & (f_2,f_1) & (f_2,f_2) \end{array} \\ \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} \end{array} \tag{3.17}$$

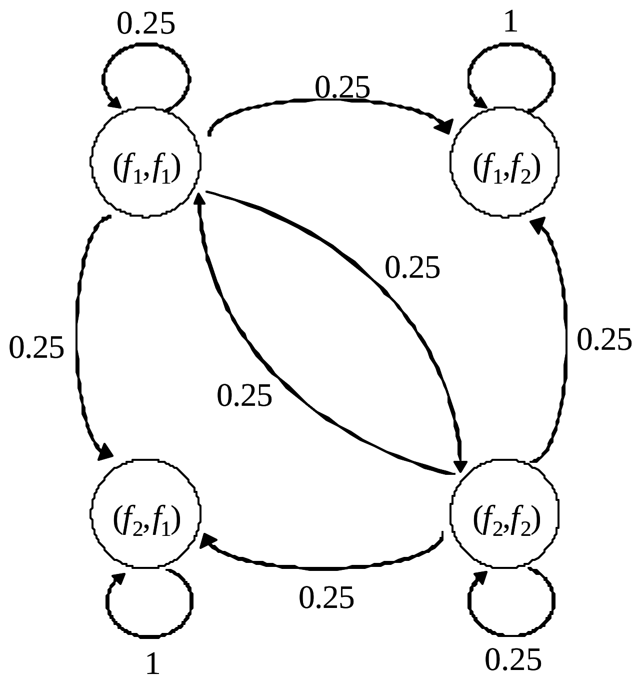

Figure 3.7: Digraph of DFS Example

.

Note that this Markov chain forms an absorbing Markov chain with (f_1,f_2) and (f_2,f_1) as absorbing states and $\{(f_1,f_1)$ and (f_2,f_2) as transient states. Thus we immediately know that this network has two steady-states $[(f_1,f_2)$ and $(f_2,f_1)]$ and that the network will converge to these states. Further, by evaluating (3.15)and (3.16) for **N**, **t**, and **B**, respectively, we can determine how long we can expect to remain in a transition state and how what the distribution of steady-states will be given an initial choice of channels.

So we know from (3.18) that with repeated trials of the network starts from (f_1,f_1), the system will on average pass through (f_1,f_1) 1.5 times and (f_1,f_1) 0.5 times, and we know from (3.19) that the system is equally likely to end up in either absorbing state.

$$\mathbf{N} = \begin{array}{c} \\ (f_1,f_1) \\ (f_2,f_2) \end{array} \begin{array}{c} \begin{array}{cc} (f_1,f_1) & (f_2,f_2) \end{array} \\ \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \end{array} \quad (3.18)$$

$$\mathbf{B} = \begin{array}{c} \\ (f_1,f_1) \\ (f_2,f_2) \end{array} \begin{array}{c} \begin{array}{cc} (f_1,f_2) & (f_2,f_1) \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \end{array} \qquad (3.19)$$

3.4 Summary

This chapter introduced several powerful techniques for analyzing the interactions of procedural cognitive radios based on the knowledge of an evolution function, d to determine steady-states, optimality, convergence, and stability. These models and the techniques for establishing if a cognitive radio network satisfies the conditions of the model are summarized in Table 3.1.

Table 3.1: Presented Models

Model (Section number)	Basic model	Identification
Dynamical Systems (3.1)	evolution equation $a\left(t^{k+1}\right)=d^t\left(a\left(t^k\right)\right)$	Assumed to exist
Contraction Mappings (3.2)	$\left\Vert f(a), f(b)\right\Vert \leq a\left\Vert a,b\right\Vert$ $\forall b,a \in A$	Blackwell's conditions
Standard Interference Function Power Control (3.2.4.1)	$d_j\left(\mathbf{p}\left(t^k\right)\right)=p_j\left(t^k\right)I_j\left(\mathbf{p}\left(t^k\right)\right)$	$I(\mathbf{p})$ satisfies positivity, montonicity, and scalability
Ergodic Markov Chain (3.3.2)	$P\left(a\left(t^{k+1}\right)=a^k \mid a(0),\ldots,a(t)\right)$ $=P\left(a\left(t^{k+1}\right)=a^k \mid a\left(t^k\right)\right)$	$\exists\, k$ such that $\mathbf{P}^k$ has all positive entries
Absorbing Markov Chain (3.3.3)	$\mathbf{P}'=\left[\begin{array}{c\|c} \mathbf{Q} & \mathbf{R} \\ \hline \mathbf{0} & \mathbf{I}^{ab} \end{array}\right]$	Apply model definition

For these models, this chapter presented analysis insights that can be gleaned by demonstrating that a procedural cognitive radio network satisfies the modeling conditions for one of the models listed in Table 3.1. The steady-state properties, the convergence properties, and the stability properties for each of these models are summarized in Table

3.2 Table 3.3, and Table 3.4, respectively. We also presented an approach to determining the desirability of network behavior –evaluation of a network objective function.

Table 3.2 Steady-State Properties by Model

Model (Section number)	Existence	Identification
Dynamical Systems (3.1)	Maybe, evaluate Leray-Schauder-Tychonoff theorem on evolution equation	Exhaustive Search, Solve $d\left(a^*\right)=a^*$
Contraction Mappings (3.2)	Yes (Banach's Theorem)	Recursion (Unique steady-state)
Standard Interference Function Power Control (3.2.4.1)	Yes ([Yates_95])	Recursion (Unique steady-state), $\mathbf{Zp}=\vec{g}$
Ergodic Markov Chain (3.3.2)	Yes (Ergodocity theorem)	Recursion (Unique distribution), Solve $p^{*T}\mathbf{P}=p^{*T}$
Absorbing Markov Chain (3.3.3)	Yes (Definition)	$p_{mm}=1$

Table 3.3 Convergence Properties by Model

Model (Section number)	Sensitivity	Rate
Dynamical Systems (3.1)	Apply Lyapunov's direct method (when possible)	No general technique
Contraction Mappings (3.2)	Everywhere convergent	$\left\|a\left(t^k\right),a^*\right\|\leq\frac{a^k}{1-a}\left\|a\left(t^1\right),a\left(t^0\right)\right\|$
Standard Interference Function Power Control (3.2.4.1)	Everywhere convergent	$\left\|\mathbf{p}\left(t^k\right),\mathbf{p}^*\right\|\leq a^k\left\|\mathbf{p}(0),\mathbf{p}^*\right\|$ $a=r\left(\mathbf{H}\right)$
Ergodic Markov Chain (3.3.2)	Converges to distribution from all starting distributions	Transition matrix dependent
Absorbing Markov Chain (3.3.3)	$\mathbf{B}=\mathbf{NR}$	$\mathbf{t}=\mathbf{N1}$

Table 3.4 Stability Properties by Model

Model (Section number)	Lyapunov Stability	Attractivity
Dynamical Systems (3.1)	Apply Lyapunov's direct method (when possible)	Apply Lyapunov's direct method (when possible)
Contraction Mappings (3.2)	Global	Global
Standard Interference Function Power Control (3.2.4.1)	Global	Global
Ergodic Markov Chain (3.3.2)	No	No
Absorbing Markov Chain (3.3.3)	Not guaranteed.	If unique absorbing state

As we saw with the Standard Interference Function, sometimes cognitive radio networks satisfy the conditions of multiple models. In these cases, the analytic insights from each of the applicable multiple models are available. While this Chapter presents a significant number of useful analytic results, the reader should be aware that this Chapter was only able to include a brief treatment of these extensive models. In fact, many of these models have entire disciplines dedicated to their analysis and application. Accordingly, the interested reader is encouraged to explore the texts listed in the references for further study.

3.5 References

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Chapter 4: Game Theory

"My work is a game, a very serious game." – M.C. Escher

By applying the techniques of the preceding chapter, we can analyze the interactions of procedural cognitive radios with single-valued decision rules where the evolution of network states are captured by the evolution function $a^{k+1} = d\left(a^k\right)$. However, for ontological cognitive radios, e.g., [Kokar_06], and procedural cognitive radios whose decision rules incorporate a degree of randomness, e.g., the genetic algorithms of [Rondeau_04], it is not generally possible to express the network behavior in terms of an evolution function as the same input may produce very different outputs.

Lacking an evolution function, the dynamical systems and the contraction mappings approaches considered in Chapter 3 will be insufficient for modeling or analyzing these systems. At least for genetic algorithms, the network could be modeled and then analyzed using Markov models. However, any useful transition matrix would have to be determined empirically – the very process we are seeking to avoid. So if we were limited to traditional engineering analysis techniques, modeling and analyzing of most cognitive radio network behavior would be impractical.

Lacking a well defined evolution function the only information we may be able to infer about the decision processes of these radios are the actions available to each radio (as governed by local policy), the goals guiding each radio, and a general assumption that each radio will act in a way the radio believes will bring it closer to its goal. Fortunately as was first noted in [Neel_02], these conditions are sufficient for modeling and analyzing cognitive radio networks with *game theory* – a collection of models and tools for analyzing interactive decision problems originally developed in economics and since extended to numerous other domains such as evolutionary biology. With game theory we are able to model and analyze cognitive radio networks using the radios' goals and only broad assumptions about the radios' decision rules, e.g., that the radios attempt to choose waveforms that the radios believe will maximize their goal given their observations of the network state. This chapter presents the insights that game theory can provide to the modeling, analysis and design of cognitive radios.

While the numerous examples in this chapter make it clear that game theory is a natural modeling technique for cognitive radios, some of the assumptions made in the development of game theoretic models and analysis techniques intended to study the interactions of humans are inappropriate for the study of the interactions of cognitive radios. Humans tend to relate their desire for different outcomes *ordinally*, i.e., we may want an apple more than an orange but we find it difficult to quantify exactly how much more. Cognitive radios are currently explicitly evaluating their outcomes *cardinally*, i.e., a radio has a defined goal that assigns numbers to different outcomes, e.g., SINR, BER, and latency, which permits a cognitive radio to quantify exactly how much more valuable one resource allocation is than another. Humans make mistakes implementing their choices, a cognitive radio will never intend to implement one waveform and implement another. However, both humans and cognitive radios make observation errors – hearing one thing when another was said or misclassifying a signal. Sometimes these differences in assumptions are inconsequential to the analysis, and sometimes small differences in assumptions lead to very different results.

While the game models and analysis techniques developed in economics will be applicable to most cognitive radio scenarios, they are sometimes inappropriate and sometimes less powerful than the techniques we introduced in Chapter 3. Because of these differences, game models and analysis techniques should not be uncritically applied to cognitive radio networks without first understanding the differences in assumptions and the analyst should be receptive to combinations of game theory and traditional engineering analysis. These differences are highlighted throughout the remainder of this chapter as they become important.

The remainder of this is organized as follows. Section 41 presents the fundamental components of game models. Section 4.2 defines the basic game models used in subsequent analysis. Section 4.3 presents basic techniques for characterizing network steady states. Section 4.4 presents the traditional game theoretic technique for ascertaining optimality. Section 4.5 shows how modeling a cognitive radio network permits the establishment of broadly applicable convergence properties. Section 4.6 considers how the presence of noise impacts the steady-states and convergence properties of cognitive radio networks from a game theoretic perspective. Throughout these sections, the analytically derived properties of cognitive radio networks are used to draw novel implications for the design of cognitive radio networks.

4.1 Basic Elements of Game Theory

To begin our discussion of game theory and the analysis of cognitive radio networks, let us formally define what we mean by the terms *game* and *game theory*.

Definition 4.1: *Game*
A *game* is a model of an *interactive decision process*.

In brief, an interactive decision process is a process whose outcome is a function of the inputs from several different decision makers who may have potentially conflicting objectives with regard to the outcome of the process. To an economics game theorist, calling a game a model of a generic interactive decision process may seem overly broad as most games are rigorously defined only for specific interactive decision processes with different game models, such as normal form games, extensive form games, and transferable utility games, applied to study specific classes of interactive decision processes.

As the model of a cognitive radio network in Chapter 2 makes subtle violations of each of the traditional game models, a more traditional and narrower definition would eliminate our cognitive radio network model. However, our cognitive radio network model preserves enough of the features of traditional game models that most traditional game theoretic results and concepts can be directly applied to our models. In light of this fact, it seems logical to adopt a more expansive definition of *game* that encompasses both the traditional game models and our cognitive radio network model.

<u>Definition 4.2</u>: *Game Theory*
Game theory is a collection of models (*games*) and analytic tools used to study interactive decision processes.

Again this definition may seem overly broad to an economics game theorist who might feel more comfortable referring to more specific classes of noncooperative or cooperative game theory. For our analysis of cognitive radio networks we leverage many of the traditional analysis tools of noncooperative game theory but also draw on the analysis techniques of Chapter 3 indicating that the broader definition is again more appropriate.

The following gives a brief description of these components in light of the cognitive radio network model we introduced in Chapter 2 and the modeling terminology applied to interactive decision processes.

4.1.1 Basic Modeling Elements of a Game

As defined in the preceding, a game is a model of an interactive decision process. Whether explicitly or implicitly, every game includes the following components:

- A set of *players*,
- *Actions* for each of the players,
- Some method for determining *outcomes* according to the actions chosen by the players,
- *Preferences* for each of the players defined over all the possible outcomes,
- *Rules* specific to the model, e.g., the order of play.

Each of the modeling elements in a game is related to a specific component of an interactive decision process as shown in Table 4.1.

Table 4.1 Relationships Between Game Elements
and Interactive Decision Process Components

Game	$\Leftrightarrow$	Interactive Decision Process
Player	$\Leftrightarrow$	Decision Maker
Actions	$\Leftrightarrow$	Inputs
Outcomes	$\Leftrightarrow$	Outputs
Preferences	$\Leftrightarrow$	Decision Maker Objectives
Rules	$\Leftrightarrow$	Decision Timings, Radio Capabilities

The following gives a brief description of these components in light of the cognitive radio network model we introduced in Chapter 2.

4.1.1.1 Players

The players are the decision making entities in the interactive decision process – the cognitive radios in the network. For notational continuity with Chapter 2, we refer to the set of players (cognitive radios) as N and individual players as i or j. As a rule, games only consider situations where there are two or more players as a single player game would by definition not be an *interactive* process. Throughout this chapter we use the terms *radios* and *players* interchangeably.

4.1.1.2 Actions and Outcomes

The actions are the adaptations available to the players where outcomes are determined by the actions and the particular system in which the players are operating. For our purposes, we continue to use actions and outcomes in the same manner as we used in Chapter 2 in that there is some function that relates action tuples to outcomes. The actions are the adaptations (waveforms) available to the radio, and the outcomes are the network states that result from each radio's (or link's) choice of waveform.

4.1.1.3 Preferences and Utility Functions

In a game, it is assumed that each player, j, has a preference relation, $\precsim_j$, that describes that player's preferences with respect to all the possible outcomes in the outcome space, O. We write $o^2 \precsim_j o^1$ if player j prefers outcome o^1 at least as much as it prefers outcome o^2; $o^2 \prec_j o^1$ if j strictly prefers o^1 to o^2; and $o^2 \sim_j o^1$ if j is indifferent between o^1 and o^2.

For example, suppose player j is faced with two possible outcomes – one where it receives apples and another where it receives oranges. If player j prefers apples to oranges, then we would write oranges $\precsim_j$ apples implying that j is at least as satisfied by receiving apples as it would be by receiving oranges; if j is indifferent between the two outcomes, we write oranges $\sim_j$ apples.

For small games, we can list all of the preference relations for every player over all possible outcomes. However, as the size of the game grows this can quickly become unwieldy. For instance, a n player game where each player has m actions could reasonably have m^n different outcomes. Accordingly, a full listing of all the preferences for a single player requires defining $\left(m^n\right)\left(m^n+1\right)/2$ preference relations; a listing for all players requires defining $n\left(m\times n\right)\left(m\times n+1\right)/2$ preference relations.

To capture these preference relations in a more compact way, game theorists frequently employ *utility functions* (sometimes called the players' objective or payoff functions) where each player assigns a real number (called the *payoff*) to each outcome (for mathematical rigor, $u_i : O \rightarrow \mathbb{R}$) in such a way that if $o^2 \prec_i o^1$ then $u_i\left(o^2\right) < u_i\left(o^1\right)$, if $o^2 \sim_i o^1$ then $u_i\left(o^2\right) = u_i\left(o^1\right)$, and if $o^2 \precsim_i o^1$ then $u_i\left(o^2\right) \leq u_i\left(o^1\right)$.

For cognitive radios, the opposite scenario exists – explicit utility functions exist and preferences must be inferred. Fortunately, it is very simple to go from utility functions to preference functions – simply use the relation that $u_i\left(o^2\right) \leq u_i\left(o^1\right) \Rightarrow o^2 \precsim_i o^1$. Even for algorithms where a clear objective is not available, if we know a cognitive radio's decision update rules, then we can infer some preference relations by examining the adaptations.[1] Assuming other radios' adaptations are fixed, if an adaptation of radio j causes a change from o^1 to o^2, then it is reasonable to assume that $o^1 \precsim_j o^2$, i.e., the algorithm "prefers" o^2 to o^1. Particularly in light of inferring preferences from the

[1] For the game theorist, we are implicitly assuming our players exhibit perfect rationality - an assumption that seems reasonable in light of the fact that our players are programmable machines.

behavior of algorithms, it can be readily seen that the concept of preferences can be applied to adaptive radios and not just to cognitive radios. Thus even the procedural radios considered in Chapter 3 can still be modeled and analyzed with game theory, though less elegantly. Still, a game theorist may feel more comfortable describing the preferences of a device that is actually aware of what it is doing.

As utility functions are really intended to capture a player's preference relation in a compact manner, the exact numbers assigned by utility functions are generally of secondary importance, assuming, of course, that the utility functions preserve the preference relations. For example, suppose player j prefers apples to oranges. From the perspective of preference relationships, $u_j(apple)=1$ and $u_j(orange)=0.5$ is equivalent to writing $u_j(apple)=1000$ and $u_j(orange)=-p^e$; faced with the choice between an apple and an orange, player j still prefers the apple and would be predicted to choose the apple. This reassignment of numbers is clearly appropriate for humans where utility functions are actually useful fictions that permit more elegant analysis than a reliance solely on preference relations. For cognitive radios, utility functions are not merely a useful fiction as cognitive radios are explicitly evaluating the benefit it receives from the network state. While this implies that we should carefully consider any similar reassignment of values for cognitive radios, we find the concept of preference preserving revaluations of utility functions useful in Chapter 5 for identifying a critical game property.

Game theorists are frequently plagued with doubts if the utility function accurately reflect a player's preferences and if the player can accurately evaluate its preferences. For instance, the inability to compare apples and oranges is part of our everyday conversation – yet the preceding not only assumed player j compared an outcome of apples to an outcome of oranges, the player was also capable of assigning numerical values to apples and oranges. If we permit fractional apples and oranges, would it be reasonable to assume a human would be able to differentiate between 3.00001 and 3 apples and be able compare the value of these two outcomes against receiving 5.3781 oranges? Fortunately, this is not a problem for the cognitive radio analyst as a radio's utility function is just the goal which informs the radio's decision process.

In game theory parlance, a player acting in its own interest (or acting in a way it believes increases its payoff) – no matter how difficult the calculation or fine the distinction in payoffs – is said to be *rational* [Osborne_94]. When we can assume rationality, these shortcomings can be ignored and our efforts can focus on the analysis. Fortunately we do assume our players are rational – an assumption that seems reasonable in light of the fact that our players are programmable machines evaluating cardinal utility functions. We also assume that the radios are acting *autonomously*. With autonomous rationality, we say that a player or its decision rule is *autonomously rational*, a concept formalized in Definition 4.3.

Definition 4.3: *Autonomously rational*[2]
A decision rule, $d_i : A \to A_i$ is said to be *autonomously rational* if $b_i \in d_i(a)$ with $b_i \neq a_i$ implies $u_i(b_i, a_{-i}) > u_i(a_i, a_{-i})$.

Conventionally, a game theorist will define utility as a function of the action space under the implicit assumption that there is a clear mapping between actions and outcomes. This allows the game theorist to remove a step in the analysis process and instead study utility functions of the form $u_i : A \to \mathbb{R}$ instead of $u_i : O \to \mathbb{R}$. For cognitive radios, we have made the explicit assumption that there exists a clear relationship between actions and outcomes so the remainder of this document expresses the radios' utility functions as functions of the action space instead of the outcome space.

As different players generally ascribe different valuations to the same action vector (or outcome), it is sometimes convenient to make use of a *payoff vector* that lists the utility that each player assigns to a particular action vector. For example, rather than writing $u_1(a) = 1$, $u_2(a) = -3$, and $u_3(a) = 4$, we could write $u(a) = (1, -3, 4)$. With this

[2] A game theorist may recognize this as strict better response rationality.

notation, it also sometimes makes sense to describe a single utility function that maps *A* into $\mathbb{R}^n$ where *n* is the number of players in the game.

4.1.1.4 Modeling Rules

Different games model different kinds of situations, where most of the differences are captured by varying the players, actions, and utility functions. However, different situations have different decision timing rules that determine when the players are allowed to "play" (choose or change an action). As with the cognitive radio network model introduced in Chapter 2, the games in this chapter adopt synchronous, asynchronous, round-robin, or randomly ordered turns of play (iterations of d^t).

Depending on the system we are modeling it may be appropriate to assume different device capabilities such as knowledge of the other radios' goals or actions (or not), perfect observations (or not), and long memories of past behavior (or not). For each of the games presented in this chapter and subsequent chapters, the rules assumed as part of the model are listed.

4.1.2 Mapping the Cognition Cycle to a Game

Fundamentally, game theory can be applied to the analysis of the interactive adaptations of any set of intelligent agents, and the cognition cycle represents the processes that go on in any intelligent being – including humans. So it is not surprising that we can establish connections between the components of a game and the cognition cycle.

Going from the cognition cycle to a game, every node in a network that implements the decision step of the cognition cycle is a player (making it a decision maker in the interactive decision process). Each radio's available adaptations form the associated player's action set, and the Cartesian product of the radios' adaptations form the action space. The cognitive radio's goal supplies a player's utility function, and the outputs of the cognitive radio's observation and orientation steps are the arguments and valuation for this utility function. Loosely, the observation step provides the player with the arguments to evaluate the utility function, and the orientation step determines the valuation of the utility function. These connections are illustrated in Figure 4.1 and summarized in Table 4.2.

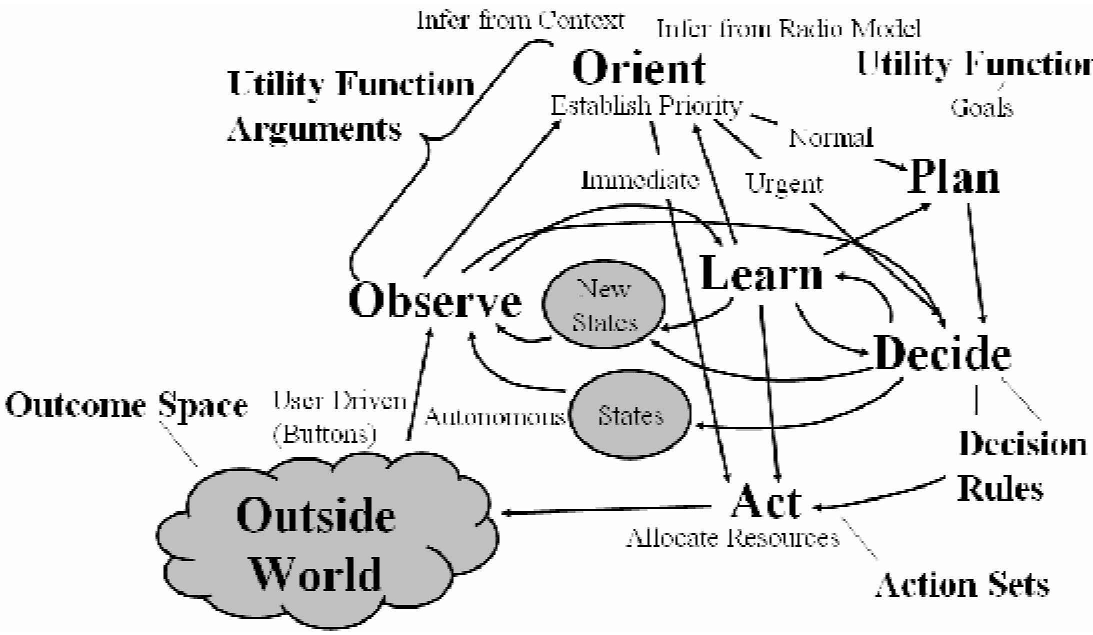

Figure 4.1: Cognition Cycle and Game Components. Modified from Figure 4.2.2 in [Mitola_00]

Table 4.2: Related Modeling Elements in a Game and a Cognitive Radio Network[3]

Game	⇔	Cognitive radio network
Player	⇔	Cognitive radio
Actions	⇔	Actions
Utility function	⇔	Goal
Outcome space	⇔	Outside world
Utility function arguments	⇔	Observations/orientation
Order of play	⇔	Adaptation timings

The observant reader may have noted that the learning step of the cognition cycle has been ignored. This is neither an oversight nor indicative of a limitation of game theory. Recall that learning of new waveforms, models, and goals is typically proposed to be performed during sleep or prayer modes [Mitola_00]. Thus these aspects of learning will

[3] An economics game theorist may be uncomfortable with observations of forming the utility function arguments as in traditional game models a one-to-one mapping exists between actions and outcomes with players generally assumed to know the relationship between actions and outcomes. In this context, the study of players' decision processes makes the most sense as a function of the players' actions. However, cognitive radios will frequently have little to no knowledge of the relationship between actions and outcomes and may not know the value of an action until they try it.

not occur while the radios are interacting and can be excluded from our modeling and analysis. Still, it does seem reasonable that in a network setting cognitive radios will be learning its environment by trying out different actions, observing the outcome, and evaluating the outcome in its goal. In fact, such an approach is proposed in [Rondeau_04]. Subsequent chapters will consider the trial and error decision processes that result when a cognitive radio is unable to perfectly predict what the resulting utility will be when it adapts its waveforms. However, the remainder of this chapter assumes perfect knowledge so that no environmental or utility function learning occurs.

4.2 Basic Game Models

Depending on the interactive decision process being modeled, a game will treat the components of players, actions, outcomes, preferences, and rules for the order-of-play in different ways. Of course, these components vary from process to process – particularly the players, actions, outcomes, and preference – but it is possible to consider broad classes of game models that are particularly useful for analyzing cognitive radio networks. The following considers three such games – the *normal form game*, the *repeated game*, and what we'll term the *myopic repeated game*.

4.2.1 Normal Form Games

The simplest and most frequently encountered game model used to describe an interactive decision making process is the *normal form game*. A normal form game assumes the following modeling rules:

- *Synchronous single-shot play* – All players make their decisions simultaneously and only make a single decision.
- *Perfect information* – The players know their own utility functions as functions of the action space and know the utility functions for all the other players in the game.
- *Perfect implementation*[4] – All players exhibit perfect implementation, i.e., no player accidentally implements action a_j^1 instead of a_j^2.

[4] The *trembling hand* game is an example of a game where perfect implementation is not assumed. This topic is addressed in Section 4.6.1.

With these rules in place, a normal form game is completed by defining the player set, action space, and utility functions. As only these three components – player set, action space, and utility functions – vary for a normal form game, the model is completely specified by the 3-tuple[5], $\Gamma = \langle N, A, \{u_j\} \rangle$, where N is the set of players, A is the action space, and $\{u_j\}$ is the set of utility functions (goals) such that each player $j \in N$ has its own utility function, $u_j : A \to \mathbb{R}$.

Particularly for two player games (i.e., $|N| = 2$), it is convenient to represent a normal form game in *matrix form*. In a matrix form representation of a two player normal form game, all of the possible action vectors in the action space are arrayed in a matrix such that player 1's actions (the first component of the action vector) are given by the rows of the matrix and player 2's actions (the second component of the action vector) are given by the columns of the matrix. Each cell in this matrix is thus determined by a unique action vector (row, column) and is filled with the payoff vector associated with the cell's action vector. The following gives two examples of normal form games represented in matrix form.

Example 4.1: Modeling a Game of Paper Rocks Scissors

Consider a game of paper-rock-scissors where winning the game is associated with a utility of 1, losing with -1, and a tie with 0.[6] This game could be expressed in matrix form as shown in Figure 4.2. For example if player 1 (the row player) played paper, *p*, and player 2 (the column player) played rock, *R*, the resulting action vector (*p*, *R*) produces a payoff vector of (1, -1) for the players.

Γ	*P*	*R*	*S*
p	(0,0)	(1,-1)	(-1,1)
r	(-1,1)	(0,0)	(1,-1)
s	(1,-1)	(-1,1)	(0,0)

Figure 4.2: Matrix Form Representation of Paper-Rock-Scissors

[5] A tuple is an ordered set. For those more familiar with programming databases, think of a tuple as a record or a structure.

[6] As the sum of the values for each payoff vector is zero, this game is an example of a *zero-sum* game.

Example 4.2: The Cognitive Radios' Dilemma

Suppose two cognitive radios are operating in the same environment and are attempting to maximize their throughput. Each radio can implement two different waveforms – one a low-power narrowband waveform, the other a higher power wideband waveform. If both radios choose to implement their narrowband waveforms – action vector (n,N) – the signals will be separated in frequency and each radio will achieve a throughput of 9.6 kbps. If one of the radios implements its wideband waveform while the other implements its narrowband waveform – action vectors (n,W) or (w,N) – then interference occurs with the narrow band signal achieving a throughput of 3.2 kbps and the wideband signal a throughput of 21 kbps. If both radios implement wideband waveforms, then each radio experiences a throughput of 7 kbps.

These waveforms can be visualized in the frequency domain as shown in Figure 4.3 and represented in matrix form as shown in Figure 4.4. Without going into the analysis of this game (presented in Section 4.3), the insightful reader may already anticipate that this algorithm tends to lead to less than optimal performance.

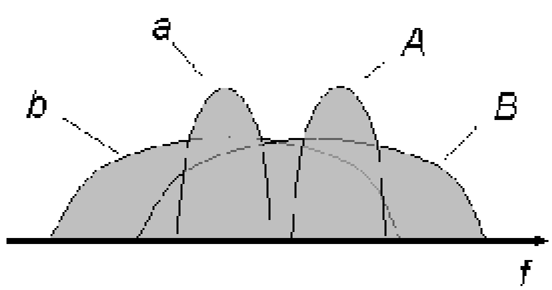

Figure 4.3: Frequency domain representation of waveforms in the Cognitive Radios' Dilemma [Neel_06]

Γ	N	W
n	(9.6,9.6)	(3.2,21)
w	(21,3.2)	(7,7)

Figure 4.4: The Cognitive Radios' Dilemma in Matrix Form [Neel_06]

4.2.2 Repeated Games

As illustrated in Figure 4.5, a repeated game is sequence of "stage games" where each stage game is the same normal form game. Due to the repetition of the normal form game, it is assumed that play is synchronous, i.e., $T_i = T_j \ \forall i, j \in N$. When the game has an infinite number of stages (e.g., Figure 4.5), the game is said to have an *infinite* *horizon*; if there are a limited number of stages, the game is said to have a *finite horizon*. Additionally, it may be the case that the game ends after a random and unknown number of stages. A randomly terminated repeated game can arise when modeling the adaptations of a mobile-assisted hand-off algorithm in cellular system where there is some nonzero probability of a radio leaving the network.

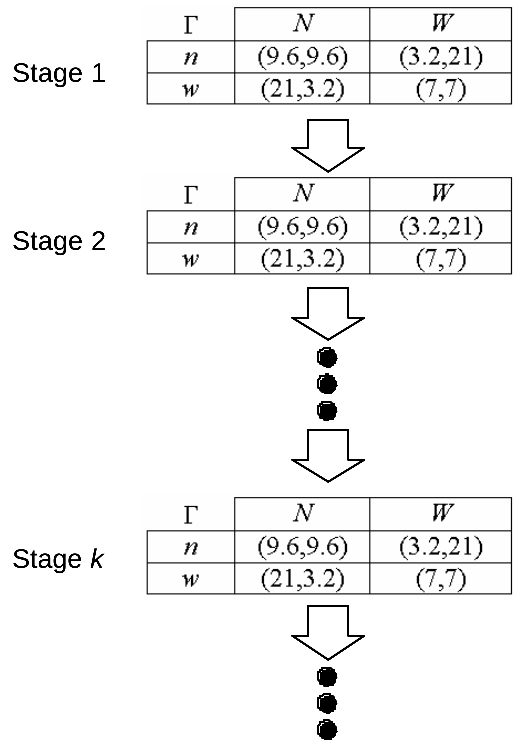

Figure 4.5 A Repeated Game is a Sequence of Stage Games

Based on their knowledge of the game – past actions, future expectations, and current observations – players choose *strategies* – a choice of actions at each stage. These strategies can be fixed, contingent on the strategies of other players, or adaptive to the actions observed in each stage. For notational continuity with Chapter 3 we denote player *i*'s strategy by the symbol d_i indicating that like the decision rules of Chapter 3, the strategy of player i determines its action in each stage of the repeated game. However, unlike in Chapter 3, it is not assumed that d_i is simply reactive to the current state of the game as a strategy may also consider past states and future expectations.

When players consider future expectations, the players employ utility functions that incorporate the payoff of the most recent stage and a time-discounted expectation of utility received from all future stages. As estimations of future values of u_i may be uncertain, many repeated games modify the original objective functions by discounting the expected payoffs in future stages by the discounting factor δ, where $\delta \in (0,1]$ such that the anticipated value in stage k to player i is given by (4.1) where a^k denotes the action vector played in stage k. Note that if δ=1, then all future payoffs are given equal weight with the present payoff.

(*Discounted payoff in stage k*) $$\tilde{u}_i\left(a^k\right) = d^k u_i\left(a^k\right) \quad (4.1)$$

Assuming all players' choices of strategies result in the sequence of action vectors (a^k), a player, i, that considers future expectations for an infinite horizon would value this sequence as shown in (4.2).

(*Expected payoff over all stages*) $$\hat{u}_i\left(\left(a^k\right)\right) = \sum_{k=0}^{\infty} d^k u_i\left(a^k\right) \quad (4.2)$$

With players considering their future payoffs, it becomes possible for players to employ strategies designed to punish players in subsequent stages after they deviated from agreed upon behavior in prior stages. When punishment occurs, players choose their actions to reduce the payoff of the offending player. This topic is revisited in Section 4.3.3 in a discussion of enforceable steady-states of repeated games.

With these components in place, we can represent a repeated game by the tuple $\langle N, A, \{\hat{u}_i\}, \{d_i\} \rangle$ where $\hat{u}_i$ evaluates the action tuple sequences generated by the strategies d_i. This is very similar to the model presented in Chapter 2 with the only differences being the following.

1) Synchronous timing is implicit to the repeated game model.
2) By choosing strategies, the players (radios) are deciding their actions for all future stages.
3) The players (radios) are weighing future payoffs as part of their decision process.

Example 4.3: Paper-Rock-Scissors Repeated Game

To illustrate the concept of a repeated game, consider the game of paper-rock-scissors illustrated in Figure 4.6 which indicates continuous repetition of the paper-rock-scissors game of Example 4.3. Figure 4.6 shows a single iteration of the game where player 1 has chosen scissors and player 2 has simultaneously chosen paper. This dictates an outcome where player 1 wins (and player 2 loses), so player 1 accrues a utility of +1 and player 2 receives a utility of -1.

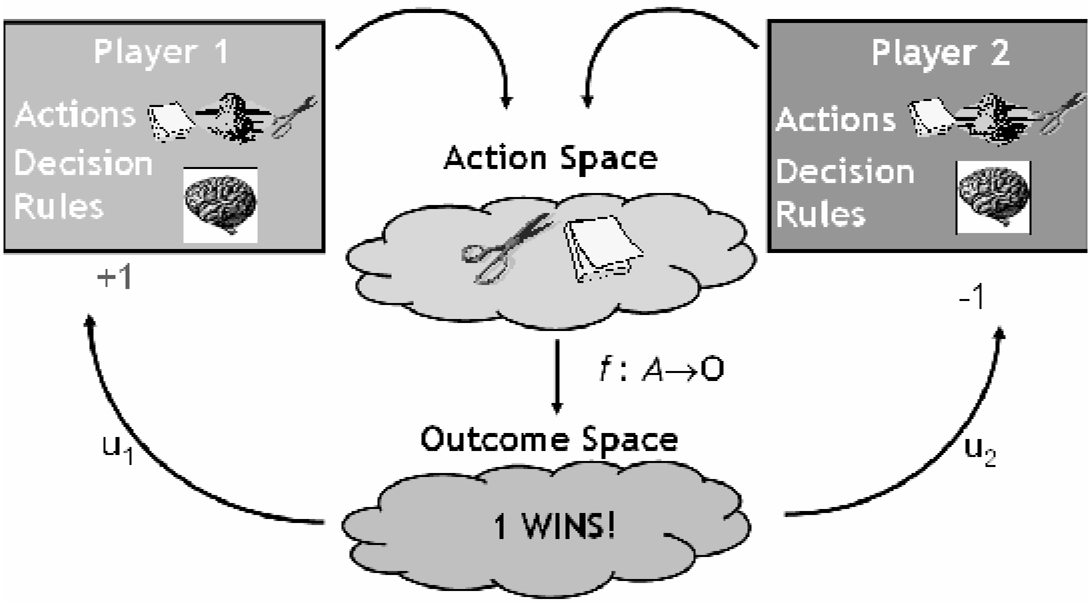

Figure 4.6: A Repeated Game of Paper-Rock-Scissors [Neel_06]

In general the players continue to adapt their decisions based on their observations about previous actions/outcomes and their expectations for future play as guided by their decision rules – perhaps based off an observation that the other player tends to prefer to play scissors. Or considering specific strategies, suppose player 1 always plays rock and player 2 alternates between paper and scissors. If the players are discounting future payoffs by a factor of 0.5 and assuming player 2 starts with paper, then player 1 would value this combination of strategies as $\sum_{k=0}^{\infty} 0.5^{2k+1}$ =0.5/(1-0.25)=2/3 and player 2 as 1/(1-0.25)=4/3.

Example 4.4: Mobile Assisted Power Control

Consider a single cell which has ten mobiles updating their power levels at a rate of 1 kHz. In the interim between power level updates, each mobile has a probability of α of leaving the network and a new mobile enters the network with probability β. If a mobile leaves the network or if a mobile enters the network, the game terminates as the players in the model have changed. So after k iterations the probability that the network is the same as when it began is given by $(1-\alpha)^k(1-\beta)^k$.

We can then represent this game as a repeated game with discounted payoffs as follows. A single power control iteration can be modeled by the normal form game $<N, A, \{u_i\}>$ where $N = \{1,2,\ldots,10\}$, A is the action (power) space, and $u_i : A \rightarrow \mathbb{R}$ represents the value that radio i assigns to the possible action tuples. When evaluating a strategy that results in the action sequence (a^k), a radio's discounted payoff in stage k is as shown in (4.1) and total expected benefit from the strategies that result in $\{a^k\}$ is as shown in (4.2) where $d = (1-a)(1-b)$.

4.2.2.1 Myopic Repeated Games

Instead of considering past actions, future expectations, and current observations, players in a repeated game may also behave *myopically*. Then we assume that there is no communication between the players, memory of past events, or speculation of future events. Any adaptation by a myopic player is based on its most recent observation of the stage game. As players make no consideration of future payoffs, complex multi-stage strategies are not possible. However, simpler myopic strategies, such as the *best response dynamic* and the *better response dynamic*, can still be employed. As these myopic strategies are reactive as in Chapter 3, d_i in a myopic repeated game can be considered equivalent to Chapter 3's decision rule. These strategies and others are discussed in Section 4.5.

Myopic repeated games can be compactly represented by the tuple $\langle N, A, \{u_i\}, \{d_i\}\rangle$ where $\{u_i\}$ is the repeated stage game utility function and where synchronous updates are assumed.

4.2.2.2 Non-Synchronous Myopic Repeated Games

As typically defined in the game theoretic literature, all decisions at each stage in a repeated game are made simultaneously. While simultaneous play certainly happens in some interactive decision problems, the implied synchronization is difficult to achieve in most cognitive radio networks so random or asynchronous decision timings are encountered more frequently. A game theorist might model such a situation as an extensive form game [Fudenburg_91], but the amount of structure inherent to a definition of an extensive form game is excessive for situations where radios retain the same actions and goals throughout the process.

Instead, this chapter uses what we term the *non-synchronous myopic repeated game*. As its name implies, a non-synchronous myopic repeated game is a repeated myopic game in which decisions do not have to be made synchronously. This has the modeling effect of limiting the number of players permitted to update its strategy at each stage, which we illustrate using the terms defined in Chapter 2 in the following.

- If modeling a process with round-robin timing, player k would get to adapt every $k|N|$ stages.
- If modeling a process with random decision timings, at each stage one player would be randomly selected and permitted to change its strategy.
- If modeling an asynchronous decision process, at each stage a random subset of players would be permitted to change their strategies.
- If modeling a synchronous decision process, the game is a myopic repeated game.

With this in mind, it is useful in subsequent discussions to consider non-synchronous myopic repeated games represented by the tuple $\langle N, A, \{u_i\}, \{d_i\}, T\rangle$. Note that these are the same modeling components considered in the model of Chapter 2.

Example 4.5: FM-AM-Spread Spectrum Repeated Game

Two cognitive radios are operating in the same environment and are attempting to independently achieve the highest possible voice quality. Each radio can implement three different waveforms - an FM, an AM, and a spread spectrum waveform. The general dynamics of this game are illustrated in Figure 4.7 where one randomly selected radio adapts its waveform in each cycle with the her radio's waveform held constant for the stage (cycle). The combination of the adaptation by one radio and the continued waveform by the other radio specify an action vector. Via the outcome function, this action vector determines an outcome. In this case, the radios observe their SINR which based on the radios' application-determined orientation and goal (voice quality) specifies a utility for that SINR. Based on their observations and inferences about the future, the radios' cognition cycle would then determine their next action in the repeated game.

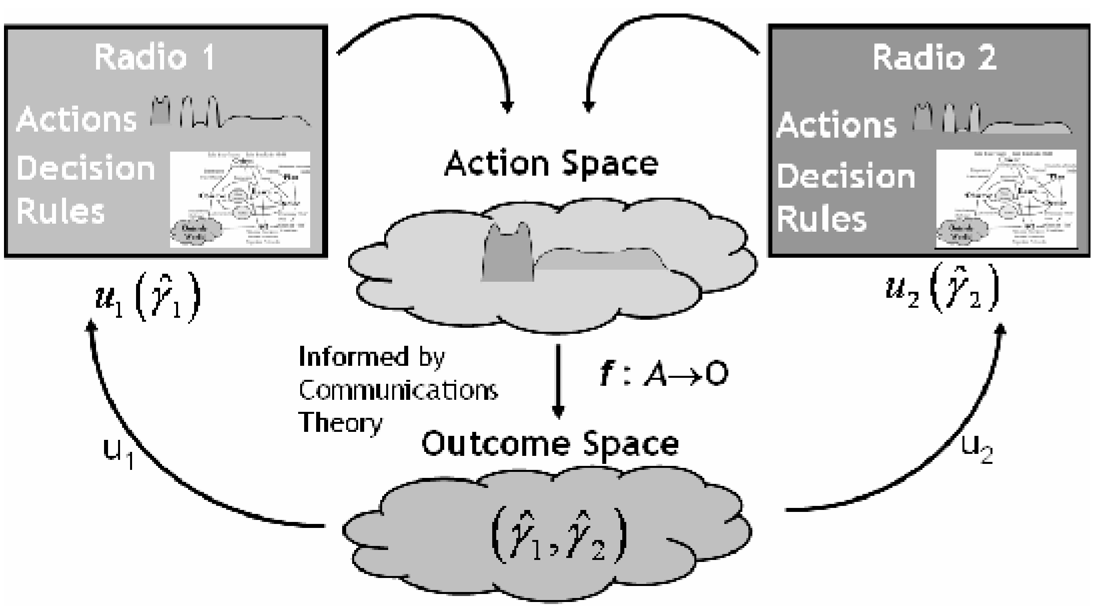

Figure 4.7: A Repeated Two Player Cognitive Radio Game [Neel_06]

4.3 Steady States

Game theory provides a number of generalizable tools for analyzing games. However, unlike the techniques described in Chapter 3, game theoretic analyses are not limited to specific decision rules. Rather they can be applied to analyze entire classes of decision rules if the radios' goals, observations, and available adaptations are known. The best example of this decision rule flexibility is the steady-state concept in game theory– the *Nash equilibrium*.

4.3.1 Nash Equilibrium

In game theory, the most frequently discussed steady-state concept is the *Nash Equilibrium* (NE). Informally, an action vector is an NE if no player (radio) can improve its performance without colluding with another player. Formally, an NE can be defined as shown in Definition 4.4.

<u>Definition 4.4</u>: *Nash Equilibrium* (NE)
An action vector , a^* is said to be an NE if and only if $u_i\left(a^*\right) \geq u_i\left(b_i, a^*_{-i}\right)\ \forall i \in N, b_i \in A_i$.

[Osborne_94] interprets an NE as *"A steady-state where each player holds a correct expectation of the other players' behavior and acts rationally."* For a normal form game where perfect knowledge can be assumed, this is a reasonable interpretation. Moving from game models to cognitive radio networks, it may be reasonable to assume that the cognitive radios know the form of the other cognitive radios' utility functions if all radios in the network have the same goal (e.g., maximizing SINR) or if the radios can poll the other radios. However, the infinite number of possible channel conditions makes it unlikely that a cognitive radio will know the precise values of other radios' utility functions. Even without any ability to infer other players utility functions, the NE concept has a significant implication for cognitive radio algorithms modeled as a repeated game.

<u>Theorem 4.1</u>: NE and Cognitive Radio Network Steady States (*)
Given cognitive radio network $\langle N, A, \{u_i\}, \{d_i\}, T\rangle$ where all players are autonomously rational, if the game $\langle N, A, \{u_i\}\rangle$ has an NE a^*, then a^* is a fixed point for *d*.
Proof: Suppose a^* is not a fixed point. Then for some $i \in N$, there must be some $b_i \in d_i\left(a^*\right)$ with $b_i \neq a_i^*$ such that $u_i\left(b_i, a_{-i}^*\right) > u_i\left(a_i^*, a_{-i}^*\right)$. But this contradicts the assumption that a^* is an NE. Therefore, a^* must be a fixed point for *d*.

So without knowing anything about the network other than the players are autonomously rational, we know that a NE must be a fixed point for all decision rules that satisfy individual rationality. This does not preclude the existence of other non-NE fixed points for certain autonomously rational decision rules, but the following lists some selected conditions for which the set of fixed points of autonomously rational decision rules are coincident with the game's NEs for round-robin, random, asynchronous, and synchronous timings.

- Best response, i.e., $d_i(a) = \{b_i \in A_i : u_i(b_i, a_{-i}) \geq u_i(a_i, a_{-i}) \forall a_i \in A_i\}$
- Random better response, i.e., $d_i(a) = rand(\{b_i \in A_i : u_i(b_i, a_{-i}) > u_i(a_i, a_{-i})\})$
- Exhaustive better response on a finite action space, i.e., if there exists a $b_i \in A_i : u_i(b_i, a_{-i}) > u_i(a_i, a_{-i})$, then $d_i(a) \in \{b_i \in A_i : u_i(b_i, a_{-i}) > u_i(a_i, a_{-i})\}$.

Example 4.6: Prisoners' Dilemma[7]

In a scene that could be "ripped from the script" of a Law & Order episode, the original formulation of the Prisoners' Dilemma considers a scenario where a serious crime and a minor crime have been committed and the police have two suspects that can be placed at the scene of the major crime and are known to have committed the minor crime. The police separate the two suspects and the district attorney (DA) independently offers each the following deal.

If both suspects deny involvement, then the DA will charge both with the minor crime and each will get one year in prison. If however, one suspect chooses to confess to both of the suspects' involvement in the major crime while the other continues to deny involvement, the one that confesses will be set free and the other will receive 15 years. Should they both choose to confess to the major crime, each will receive 10 years in prison.

This situation can be visualized as a game as shown in Figure 4.8 where the Nash Equilibrium for this game is for both prisoners to confess – (c,C) as neither suspect would choose to deviate from confessing as to do so would increase their individual prison sentence from 10 to 15 years.

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[7] The introduction to this example is taken from [Neel_02].

Γ	*C*	*D*
c	(1,1)	(0,15)
d	(15,0)	(10,10)

Figure 4.8: Prisoners' Dilemma Matrix Form Representation

To give the game a more general setting, the prisoners' dilemma can be abstractly defined using the 2×2 symmetric game matrix shown in Figure 4.9 where $y < z < w < x$. At its core, the prisoners' dilemma models a situation frequently encountered in real life – where it is easy to put individual interests above the good of the group. If both players cooperate, the social maximum is obtained, but the incentives are such that both players see an individual benefit from not cooperating so the social minimum is obtained.

Γ	*C*	*D*
c	(w, w)	(x, y)
d	(y, x)	(z, z)

Figure 4.9: Abstract Representation of the Prisoners' Dilemma Game Matrix

As such, this general model of the Prisoners' Dilemma is *the* classic problem in game theory as economists attempt to develop additional structures such that the model's steady-state aligns with the NE. The Prisoners' Dilemma has been applied to the Cold War, trade negotiations, the game show *Friend or Foe*, a "team building" exercise I participated as a co-op at Nortel, and even has an entire book dedicated to applications of the problem [Poundstone_92]. One of the more commonly discussed methods for bringing behavior closer to the social optimum is to extend the interaction between players in the context of a repeated game which we consider in Section 4.3.3.

Example 4.7: Identifying the NE of Cognitive Radios' Dilemma

Consider a systematic application of Definition 4.4 to the Cognitive Radios' Dilemma of Example 4.2 whose matrix form representation is reproduced in Figure 4.10. For action vector (n,N), either radio can improve its performance by choosing a wideband waveform (21>9.6). For action vector (n,W) or action vector (w,N), the radio with the narrowband waveform can improve its performance by changing to a wideband waveform (7 > 3.2). For action vector (w,W), neither radio can improve its performance by switching to a narrowband waveform (3.2 < 7), thus (w, W) is identified as an NE for the game.

Γ	*N*	*W*
n	(9.6,9.6)	(3.2,21)
w	(21,3.2)	(7,7)

Figure 4.10: The Cognitive Radios' Dilemma. This game has a unique NE at (*w*, *W*).

Note that (*n*,*N*) would actually yield superior performance for both radios, but reaching (*n*,*N*) from (*w*,*W*) requires both radios to deviate from (*w*,*W*) which neither radio has a unilateral incentive to do (3.2 < 7) so the adaptation would fail the individual rationality condition. Thus the game's NE is a steady state for all autonomously rational decision rules. Also note that this game satisfies the conditions of the abstract Prisoners' Dilemma of Figure 4.9. In general most of the popular game models developed in economics settings have analogous applications in cognitive radio networks. Throughout the remainder of this chapter and subsequent chapters, connections between cognitive radio networks and traditional game models are noted and exploited.

4.3.1.1 NE Identification

As was the case for the controls theory approach, identifying the steady states of a general normal form or repeated game can be quite difficult. As the only generally applicable approach to identifying a game's NE is to perform an exhaustive search with repeated application of Definition 4.4 (as we performed in Example 4.7), NE identification for a game is an NP-complete problem [Neel_04a]. As we will see in Chapters 5 and 8, it is possible to reduce this search process for some special games, but not all games satisfy the special properties of the games in Chapters 5 and 8. Alternately, some analysts are forced to turn to simulations – the very step we're intent on minimizing. For example, [Ginde_03] used an exhaustive simulation that ran for days to show that a GPRS network employing joint rate-power adaptations had four NEs, even though the modeled system included only 7 players.

For finite games, one valuable algorithm for identifying NE is the *Iterative Elimination of Dominated Strategies* (IEDS) where a dominated strategy is defined in Definition 4.5.

Definition 4.5: *Dominated Strategy*
An action (strategy) a_i is said to be *dominated* by action b_i if $u_i(b_i, a_{-i}) \geq u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$ and $u_i(b_i, b_{-i}) > u_i(a_i, b_{-i})$ for some $b_{-i} \in A_{-i}$.

In subsequent discussions the following related terms are used:

- *Undominated Strategy* – an action (strategy) a_i for which there exists no action $b_i \in A_i$ that dominates a_i.
- *Dominant Strategy* - an action (strategy) a_i that dominates all $b_i \in A_i \setminus a_i$.

With these terms in mind, an algorithm can be defined for identifying NE in normal form game $\Gamma = \langle N, A, \{u_i\} \rangle$.

Algorithm 4.1: Iterative Elimination of Dominated Strategies (IEDS)
1. Assign k=0, $A^k = A$,.
2. k:= k+1
3. Form A^k by removing all dominated strategies (actions) $\forall\ i \in N$.
4. If $A^k = A^{k-1}$, terminate the algorithm. Otherwise return to step 2.

If Algorithm 4.1 terminates with A^k as a single action vector a^*, then a^* is an NE ([Dutta_99], Proposition 5.3]) and Γ is said to be *IEDS solvable* or *dominance solvable*. An example of an IEDS solvable game is given in the Cognitive Radios' Dilemma of Example 4.2 where after a single iteration the algorithm terminates in the action tuple (w,W) which was identified as an NE for the game in Example 4.7.

However, not all games are IEDS solvable as seen in the paper-rock-scissors game of Example 4.1. Further not all games with an NE are IEDS solvable. For instance, consider the channel selection game presented in Figure 4.11 which has two NE – (Chan. 1, Chan. 2) and (Chan. 2, Chan. 1) – and no dominated strategies.[8]

Γ	Chan. 1	Chan. 2
Chan. 1	(-1,-1)	(1,1)
Chan. 2	(1,1)	(-1,-1)

Figure 4.11: A Channel Selection Game

[8] Loosely, this game can be considered to be modeling a scenario where two radios are performing dynamic frequency selection and each radio is seeking to minimize the interference it experiences from the other radio.

Even when a game is IEDS solvable, its solution may not be the only NE in the game. For example consider the costless channel construction game illustrated in matrix form in Figure 4.12.[9] This game has two NE – (Create, Create) and (Idle, Idle) – yet Create dominates Idle for both players so Algorithm 4.1 only yields the NE (Create, Create).

Γ	Create	Idle
Create	(1,1)	(0,0)
Idle	(0,0)	(0,0)

Figure 4.12: A Channel Construction Game

Some of these limitations are removed if instead of using dominance, we use *strict dominance* as defined in Definition 4.6.

Definition 4.6: *Strictly Dominated Strategy*
An action (strategy) a_i is said to be *strictly dominated* by action b_i if $u_i(b_i, a_{-i}) > u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$.

Leveraging earlier terminology, we consider a strategy to be *strictly dominant* if it strictly dominates all other strategies and we can define an algorithm by requiring that strictly dominated strategies are removed at each iteration.

Algorithm 4.2: Iterative Elimination of Strictly Dominated Strategies (IESDS)

1. Assign k=0, $A^k = A$,.
2. k:= k+1
3. Form A^k by removing all strictly dominated strategies (actions) $\forall\ i \in N$.
4. If $A^k = A^{k-1}$, terminate the algorithm. Otherwise return to step 2.

If the revised algorithm terminates with A^k as a single action vector a^*, then a^* is still an NE, but of more significance, the NE is the unique NE for the game.

An example of an IESDS solvable game is given in the Cognitive Radios' Dilemma of Example 4.2 whose unique NE, (w,W), results from application of Algorithm 4.2. Note that if Algorithm 4.2 is applied to the Channel Construction Game of Figure 4.12, the algorithm would fail as no strategy strictly dominates any other strategy. Also of value,

[9] This game can be viewed as modeling a scenario where a pair of radios would prefer to create a communications channel, but experience no penalty if they fail to create the channel.

Algorithm 4.2 can be modified so that rather than proceeding synchronously, any order of elimination is employed ([Dutta_99], p. 59]).

4.3.1.2 NE Existence

As we saw in the preceding, not all games with NE are IEDS solvable and not all games have NE. Consider the game of paper-rock-scissors we presented in Example 4.1. An exhaustive application of Definition 4.4 will reveal that none of the action tuples are NE (indeed if any were, the game would not be very fun to play). So not only may we need to simulate for days to find the NE of a game, there might not even be any NE to find. So before searching for NE it is good to know if an NE exists.

Similar to showing that a dynamical system has a steady-state, fixed point theory can show that a game has an NE. Define a set-valued *best response function* for player *i*, $\hat{B}_i(a)$, which returns the set of actions that maximize the utility for player *i* for a given action vector *a*. This is written more formally as shown in (4.3).

(*Best Response*) $$\hat{B}_i(a) = \left\{ b_i \in A_i : u_i\left(b_i, a_{-i}\right) \geq u_i\left(a_i, a_{-i}\right) \forall a_i \in A_i \right\} \qquad (4.3)$$

Now define the synchronous best response function for all players in the game, $\hat{B}(a)$ as the simultaneous application of (4.3) $\forall i \in N$ as shown in (4.4).

(*Synchronous Best Response*) $$\hat{B}(a) = \underset{i \in N}{\times} \hat{B}_i(a) \qquad (4.4)$$

Now consider an action vector a^* such that $a^* \in \hat{B}(a^*)$. Examining Definition 4.4, we see that a^* must be an NE. So if $\hat{B}(a)$ has a fixed point, then the game has an NE. Determining that $\hat{B}(a)$ has a fixed point requires the introduction of Kakutani's fixed point theorem.

Theorem 3.2: *Kakutani's Fixed Point Theorem* [Osborne_94]
Let $f : X \rightarrow X$ be an upper semi-continuous convex valued correspondence from a non-empty compact convex set $X \subset \mathbb{R}^n$, then there is some $x^* \in X$ such that $x^* \in f(x^*)$

While a formal proof of this theorem is too long for inclusion here, that an upper semi-continuous convex valued correspondence on a compact convex set implies the existence of a fixed point can be visualized in two dimensions as shown in Figure 4.13. where the convexity and upper semi-continuity of the function ensure that the function must intersect the line $x = f(x)$.

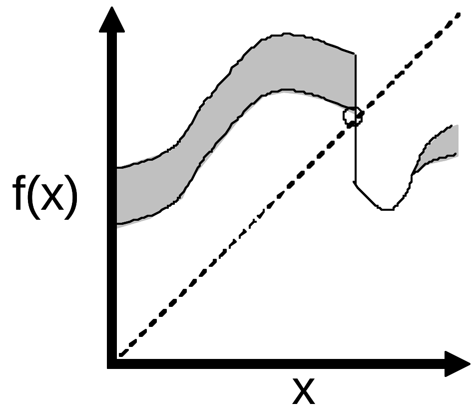

Figure 4.13 Visualization of Kakutani's Fixed Point Theorem in Two Dimensions.

Assuming we know that $\hat{B}(a)$ is an upper semi-continuous function and we know that A is a non-empty, compact and convex subset of Euclidean space, we know that the game has an NE. While verifying that A is a non-empty, compact and convex subset of Euclidean space is rather trivial, verifying that $\hat{B}(a)$ is an upper semi-continuous function without solving for $\hat{B}(a)$ appears to be rather difficult. Fortunately, Theorem 1.2 in [Fudenberg_91] provides the following theorem.

Theorem 4.3: *Glicksberg-Fan-Debreu Fixed Point Theorem* [Fudenberg_91]
Given normal form game $\Gamma = \langle N, A, \{u_i\} \rangle$ where A_i are nonempty compact convex subsets of $\mathbb{R}^m$ $\forall i \in N$. If $\forall i \in N$ u_i is continuous in a and quasi-concave in a_i then Γ has a pure strategy NE.

As alluded to in the preceding, the proof of this theorem rests on Kakutani's fixed point theorem and that upper semi-continuous $\hat{B}(a)$ is implied by all being u_i continuous in a and quasi-concave in a_i where quasi-concavity is formally defined as shown in Definition 4.7.

Definition 4.7: Quasi-concavity

A function $f : X \to \mathbb{R}$ is said to be *quasi-concave* if $\forall x^1, x^2 \in X$, $a \in (0,1)$ the following relationship is satisfied: $f\left(a\, x^1 + (1-a)\, x^2\right) \geq \min\left\{ f\left(x^1\right), f\left(x^2\right)\right\}$.

The function f would be *strictly quasi-concave* if $f\left(a\, x^1 + (1-a)\, x^2\right) >$. $\min\left\{ f\left(x^1\right), f\left(x^2\right)\right\}$. An equivalent definition of a quasi-concave function is a function for which all of its *upper level sets* are convex. Given a point a^* and a function $f : A \to \mathbb{R}$, the upper level set for a^* is given by $U\left(a^*\right) = \left\{ a \in A \colon f(a) \geq a^* \right\}$.

Contrasting quasi-concavity with the concepts of concavity and pseudo-concavity considered in Chapter 3, Figure 4.14 provides an example of a function that is quasi-concave, but is neither concave nor pseudo-concave. Lack of concavity can be verified by noting that a line between a^0 and a^2 contains points above $f(a)$. Lack of pseudo-concavity can be verified by noting that $f(a)$ is not differentiable at a^2. Relating these three concepts, all concave functions are pseudo-concave, and all pseudo-concave functions are quasi-concave. It could also be noted that this illustrated function is strictly quasi-concave (Definition 4.7, but with strict inequality) and continuous which implies that the arg max of f is unique.

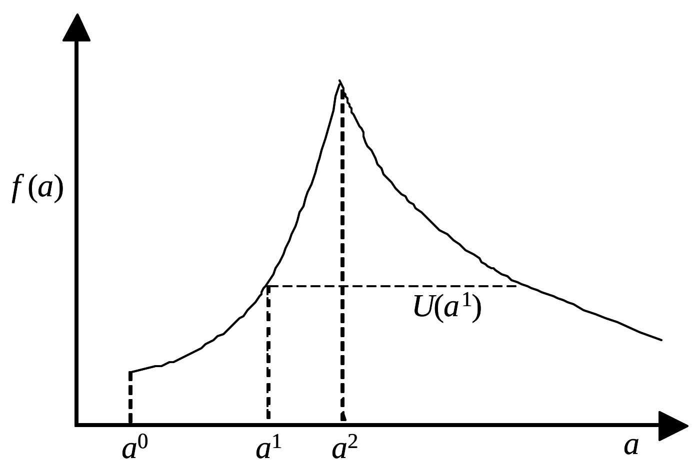

Figure 4.14: A function that is quasi-concave but neither concave nor pseudo-concave.
From Figure 15.3-15 in [Neel_06a]

There are a number of limitations to Theorem 4.3. If u_i is neither continuous in a nor quasi-concave in a_i, then Theorem 4.3 cannot be applied. Further if A is finite – a more commonly encountered condition, then Theorem 4.3 does not apply. In general, a game with a finite action space cannot be assumed to have an NE – see the paper-rock-scissors game considered in Example 4.3. However, Chapters 5 and 8 introduce additional structure which assures the existence of an NE even for finite A.

Example 4.8 Existence of a NE in a Power Control Game

In a power control game, radios adjust their power levels in an attempt to maximize some utility function, typically some function that balances SINR or throughput against power consumption or battery life. For this example, consider the power control algorithm presented in [Goodman_00] which considers a single cell where the mobiles are adapting their transmit power levels in an attempt to maximize the utility function given in (4.5)

$$u_i(\mathbf{p}) = \frac{R}{p_i}\left(1 - e^{-0.5 g_i}\right)^L \qquad (4.5)$$

which is an expression of throughput for a FSK waveform divided by transmit power p_i.

In this expression, throughput is a function of the data rate, R, the packet length, L, and the received SINR of player i's signal, g_i, where g_i is calculated as shown in (4.6)

$$g_i = \frac{W}{R}\frac{g_i p_i}{\sum_{k \in N \setminus i} g_k p_k + s} \qquad (4.6)$$

where W is the bandwidth of the transmitted signal, g_k is the gain of the k^{th} mobile to the base station, p_k is the transmit power of mobile k and σ is the noise power at the base station. This can then be modeled as a normal form game by the tuple $\langle N, \mathbf{P}, \{u_i\} \rangle$ where $\mathbf{P}$ is the power (action) space formed by the Cartesian product of the sets of power levels available to each player i, $P_i \subset \mathbb{R}$.

For the purposes of this example, assume that each P_i is compact and convex. Then comparing this game to the conditions in Theorem 4.3, it is seen that the action sets are nonempty compact convex subsets of $\mathbb{R}$ $\forall i \in N$, and that u_i is continuous in $\mathbf{p}$ $\forall i \in N$ where $\mathbf{p}$ is the power vector formed by each radio i choosing a power level from P_i. To verify that u_i is quasi-concave in p_i consider the sketch of the shape of u_i shown in Figure 4.16 (the exact values of u_i would be a function of $\mathbf{p}$ and the link gains, g_k). For this shape, any upper level set is convex – an example of which is given by the upper level set of u_i evaluated at $p_{i,1}$, $U(p_{i,1})$.

Thus the utility function is also quasi-concave and by Theorem 4.3 this game has at least one pure strategy NE and by Theorem 4.1 this NE must be a fixed point for all autonomously rational decision rules.

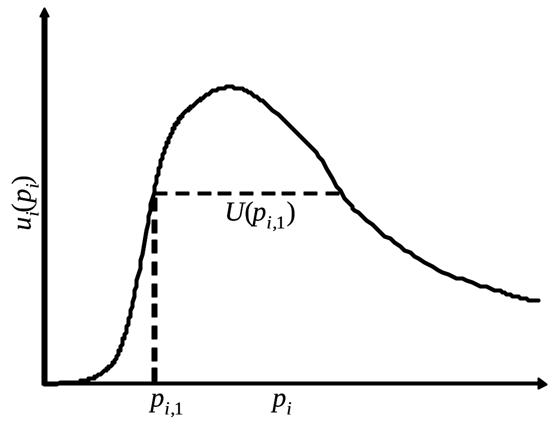

Figure 4.16: General Shape of Utility Function Given in (4.5).

When a game's utility functions are strictly quasi-concave, the best response function given in (4.3) becomes a single valued function (as opposed to set valued). Particularly when coupled with continuous and differentiable utility functions on a compact convex space, this implies that we can identify NE by simultaneously solving (4.7) $\forall\ i \in N$. While this is very similar to the fixed point identification process of Chapter 3 (simultaneous solution of fixed points of the evolution functions), solving this system of equations identifies fixed points for all self-interested autonomous decision rules.

$$\hat{B}_i(a) = a_i \tag{4.7}$$

Example 4.9: Cournot Oligopoly and Bandwidth Selection

The general Cournot Oligopoly consists of a set of n firms, all of which are producing an identical commodity, with each firm, i, free to determine the quantity of the commodity it produces, $q_i \in [0,\infty)$. Guiding this decision, each firm is attempting to maximize its profit given in (4.8)

$$u_i(q) = P(q)q_i - C_i(q_i) \tag{4.8}$$

where $P(q)$ is the price of the commodity determined by the quantity produced by all firms and $C_i(q_i)$ is firm i's cost for producing q_i units of the commodity. In general, $P(q)$ decreases as the total number of commodity units are produced increases and C_i increases with increasing q_i with the price and cost functions frequently approximated as linear functions. For instance [Fudenberg_91] gives (4.9) and (4.10) as functions for price and cost which when substituted into (4.8) yields (4.11).

$$P(q) = \max\left(0, 1 - \sum_{i \in N} q_i\right) \tag{4.9}$$

$$c_i(q) = cq_i \tag{4.10}$$

$$u_i(q) = \left(1 - \sum_{k \in N} q_k\right) q_i - c(q_i) \tag{4.11}$$

As with the Cognitive Radios' Dilemma, it is frequently possible to quickly construct a scenario in a wireless network based on a well-known scenario from game theory. Suppose a network consists of five cognitive radios with each radio, i, free to determine the number of simultaneous frequency hopping channels the radio implements, $c_i \in [0,\infty)$. Guiding this decision, each radio is attempting to maximize the difference between a function of goodput and power consumption as given in (4.12)

$$u_i(c) = P(c)c_i - C_i(c_i) \tag{4.12}$$

where $P(c)$ is the fraction of symbols that are not interfered with (making $P(c)c_i$ the goodput for radio i) and $C_i(c_i)$ is radio i's cost for supporting c_i simultaneous channels. In general, P decreases as the total number of channels implemented increases and C_i increases with increasing c_i (more bandwidth implies more processing resources implies more power consumption). If we approximate these effects as linear functions, we can rewrite (4.12) as (4.13)

$$u_i(c) = \left(B - \sum_{k \in N} c_k\right) c_i - Kc_i \tag{4.13}$$

where B is the total bandwidth that the waveforms are hopping over, K is the cost of implementing each channel, and N is the set of cognitive radios. Comparing (4.11) and (4.13), we see that this cognitive radio game is just a simple reformulation of a Cournot oligopoly. Given (4.13), the best response for radio i is given by (4.14).

$$\hat{B}_i(c) = \left(B - K - \sum_{k \in N \setminus i} c_k \right) / 2 \tag{4.14}$$

Simultaneously solving (4.7) with the best response for each radio by (4.14) yields the following system of equations.

$$\begin{array}{rcrcrcrcrcl}
c_1 & + & 0.5\,c_2 & + & 0.5\,c_3 & + & 0.5\,c_4 & + & 0.5\,c_5 & = & (B\text{-}K)/2 \\
0.5\,c_1 & + & c_2 & + & 0.5\,c_3 & + & 0.5\,c_4 & + & 0.5\,c_5 & = & (B\text{-}K)/2 \\
0.5\,c_1 & + & 0.5\,c_2 & + & c_3 & + & 0.5\,c_4 & + & 0.5\,c_5 & = & (B\text{-}K)/2 \\
0.5\,c_1 & + & 0.5\,c_2 & + & 0.5\,c_3 & + & c_4 & + & 0.5\,c_5 & = & (B\text{-}K)/2 \\
0.5\,c_1 & + & 0.5\,c_2 & + & 0.5\,c_3 & + & 0.5\,c_4 & + & c_5 & = & (B\text{-}K)/2
\end{array}$$

Simultaneously solving this set of equations yields the symmetric solution shown in (4.15). Generalizing this result to $|N|$ radios, the unique NE is given by (4.16).

$$\hat{c}_i = (B - K)/6 \ \forall i \in N \tag{4.15}$$

$$\hat{c}_i = (B - K)/(|N| + 1) \ \forall i \in N \tag{4.16}$$

4.3.2 Mixed Strategy Equilibria

To overcome the limitation that finite normal games, such as paper-rock scissors, may not have an NE, many authors have suggested the use of the mixed extension to a normal form game. In the mixed extension to a normal form game players employ "mixed" (probabilistic) strategies in the place of discrete actions.

Definition 4.8: Mixed Strategy
Given (pure) action set A_i, a *mixed strategy* for player i, $\mathbf{a}_i = \left(p_i(a_i^1), p_i(a_i^2), \ldots, p_i\left(a_i^{|A|}\right) \right)$, is an assignment of probabilities, $p_i(a_i^k)$ to each $a_i^k \in A_i$ such that $p_i(a_i^1) \in [0,1]$ and $\sum_{k=1}^{|A_i|} p_i(a_i^k) = 1$.

Those $a_i^k \in A_i$ for which $p_k > 0$ are said to be in the *support* of $\mathbf{a}_i$. Given action space A and player set N, the set of all possible mixed strategies for player i is denoted by the symbol $\Delta(A_i)$ and $\Delta(A) = \times_{i \in N} \Delta(A_i)$ is used to refer to the set of all possible mixed strategy tuples where $a_i \in \Delta(A_i)$. To complete the terminology necessary for the mixed extension of normal form game $\Gamma = \langle N, A, \{u_i\} \rangle$, given mixed strategy vector $\alpha_i \in \Delta(A_i)$, player i has an expected utility, $U_i(a)$ given by (4.17) where $p(a) = \times_{i \in N} p_i(a_i)$.

$$U_i(a) \equiv \sum_{a \in A} p(a) u_i(a) \tag{4.17}$$

With the preceding terminology in mind, given normal form game, $\Gamma = \langle N, A, \{u_i\} \rangle$, its mixed extension is given by $\Gamma' = \langle N, \Delta(A), \{U_i\} \rangle$, the mixed best response for player i to mixed strategy vector α is shown in (4.18), the synchronous best response to strategy vector α is shown in (4.19).

$$\hat{B}_i^a(a) = \{ b_i \in \Delta(A_i) : U_i(b_i, a_{-i}) \geq u_i(a_i, a_{-i}) \forall a_i \in \Delta(A_i) \} \tag{4.18}$$

$$\hat{B}^a(a) = \times_{i \in N} \hat{B}_i^a(a) \tag{4.19}$$

[Osborne_94] (p. 32) states that (4.17) is a *multilinear* function which implies that (4.17) is both continuous and concave (and thus quasi-concave) which implies that (4.19) is upper-semicontinuous. This insight sets the stage for Nash's fixed point theorem.

Theorem 4.4: *Nash's Fixed Point Theorem* ([Fudenberg_91] Theorem 1.1)
Every normal form game $\Gamma = \langle N, A, \{u_i\} \rangle$ has a where A_i are nonempty compact convex subsets of $\mathbb{R}^m$ $\forall i \in N$. If $\forall i \in N$ u_i is continuous in a and quasi-concave in a_i then Γ has a pure strategy NE.

Example 4.10: Paper-Rock-Scissors in Mixed Strategies

Consider a mixed extension to the paper-rock-scissors game presented in Example 4.1 with player 1's probabilities of playing paper, rock, or scissors as p_1^p, p_1^r, p_1^s, respectively and player 2's probabilities of playing paper, rock, or scissors as p_2^p, p_2^r, p_2^s, respectively. Due to the symmetry in this problem, the best response for player 1 is a_1^*=(p_2^r, p_2^s, p_2^p) and for player 2 is a_2^*=(p_1^r, p_1^s, p_1^p). This situation yields a unique simultaneous solution for α^* of a_1^* = (1/3, 1/3, 1/3) and a_2^*= (1/3, 1/3, 1/3).

Thus an NE exists in mixed strategies for the game of paper-rock-scissors even though no NE existed in pure strategies.

4.3.3 Enforceable Equilibria in Repeated Games

In a myopic repeated game, it is reasonable to assume that the NE of a stage game will be the repeated game's steady-state. However, if players are not myopic and are instead choosing their actions for each stage in an attempt to maximize the weighted sum of present and future payoffs as given in (4.2) and if the players incorporate the correct strategies, then many more equilibria are possible.

Those who have studied psychology know that with a properly designed regimen of "carrots" (rewards) and "sticks" (punishments), almost any behavior can be coaxed out of any animal or person. Mice can be trained to push levers and run mazes for food; dogs can be trained to act as seeing eye guides for the blind; and someday I may even learn how to cook.

In the context of a repeated game, players may also seek to shape the behavior of other players. Here, the "carrots" are choices of actions in stages that increase another player's utility and the "sticks" are choice of actions in stages that decrease another player's utility. As frequently is the case in real life, there are limits to how much punishment and how much reward can be given to other players. Suppose all players in a repeated game conspire to minimize the payoff of player i. Having control over its own action, player i can ensure that it still receives some minimum payoff $\underline{v}_i$.

For instance, consider a repeated game with the Cognitive Radios' Dilemma as the stage game. If the row player wanted to punish the column player, the row player could play w which yields the smallest possible utility for the column player ($3.2 < 9.6$, $7 < 21$). However the row player cannot actually force the column player to a throughput of 3.2 kbps. So for a single stage game, no player can be punished with a throughput less that 7 kbps. If the punishment were extended to all future stages in an infinite horizon game, then the expected payoff over all stages would be $7\sum_{k=0}^{\infty} d^k = 7/(1-d)$.

Γ	*N*	*W*
n	(9.6,9.6)	(3.2,21)
w	(21,3.2)	(7,7)

Figure 4.17: Cognitive Radios' Dilemma

If the row player demanded the column player play any sequence of actions that resulted in the column player receiving more than 7/(1-δ), it would be rational for the column player to play that sequence of actions as long as the column player believed that the punishment is credible.

Theorem 4.5: Folk theorem [Fudenberg_91]
In a repeated game with an infinite horizon and discounting, for every feasible payoff vector $v > \underline{v}_i$ for all $i \in N$, there exists a $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$ there is a steady-state with payoffs v.

To generalize the Folk theorem, given a discounted infinite horizon repeated game and through the proper choice of punishment strategies and discounting factor, δ, nearly any behavior can be designed to be the "steady-state" of the game, including sequences of actions that vary at each stage. Many authors (e.g., [Dutta_01]) have remarked that this is an unsatisfactory result because it permits too many equilibria (an infinite number, to be precise). However, this author argues that this condition merely reflects reality as evidenced by animal acts in circuses, dog shows, and cats that use toilets – all of which were trained using strategies of punishment and reward.

Making punishments credible is quite difficult in finite horizon games and is clearly impossible in a "one-off" single stage game. However, for games that repeat several times, though not infinitely, reward and punishment strategies can still improve performance though this can be quite difficult. Because of this, many algorithm designers choose situations where the players would believe the game to have an infinite horizon – an assumption we made in Chapter 2 where we wrote that T contains elements that extend to ∞.

Example 4.11 Punishment and Power Control

[MacKenzie_01] considers a single cell power control algorithm where each mobile believes it is operating in an infinite horizon game and has an utility function of the form shown in (4.5) which we repeat in (4.20). As shown in Example 4.8, the stage game has an NE. However, the NE power vector is suboptimal as lower transmit powers would yield a higher utilities for all of the mobiles.

$$u_i(\mathbf{p}) = \frac{R}{p_i}\left(1 - e^{-0.5 g_i}\right)^L \tag{4.20}$$

When the radios only view their adaptations as a single stage, instead of an iterated sequence of adaptations, the network remains in the suboptimal equilibrium. However, if the radios incorporate future payoffs into their decision making process, then the Folk Theorem assures us that many other equilibria are possible and since all devices could perform better, it should be possible to construct a reward/punishment strategy that moves the network to a better equilibrium.

[MacKenzie_01] adopts this approach and proposes that whenever a radio deviates from the optimal power vector, all other radios "punish" the offending radio by transmitting at maximum power for the duration of the next packet. This drives the offending radio's throughput to near zero thereby offsetting throughput gains that may be made by violating the agreed operating point as it is assumed that the discounting factor is very close to 1. The improvement in utilities that results from all radios implementing and being aware of this punishment strategy is shown in Figure 4.18.

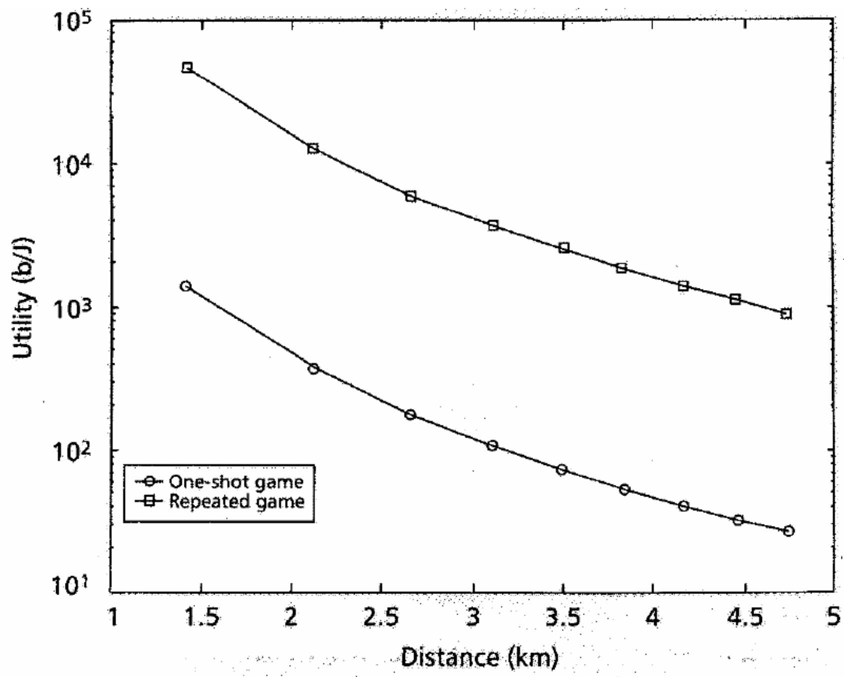

Figure 4.18 Improvement in Utilities from Enforcing a non-NE Equilibrium.
From Fig. 6 in [MacKenzie_01].

While Example 4.11 employed a punishment strategy that applied maximum punishment in the stage subsequent to deviation, there are an infinite variety of punishment schemes that could be employed. Some of the most popular strategies are listed in the following where it is assumed that player k is the deviating player and player j is the punishing player.

- *Grim Trigger* – Once player k deviates, j implements the action that minimizes k's payoffs for all subsequent stages. Because this punishment carries the threat of the least possible payoff, it permits the greatest number of equilibria. However, it also minimizes the payoff of j if implemented so some automated mechanism is generally needed to make the punishment credible.
- *Tit for Tat* – More formally defined for the Prisoners' Dilemma, with a tit-for-tat strategy, deviations by k are met by deviations by j. After k returns to the agreed strategy, j returns to the agreed strategy.
- *Generous Tit for Tat* – Rather than immediately triggering punishment, deviation is permitted to continue for a number of iterations before punishment continues.

After a number of iterations where k has returned to the agreed strategy, j also returns.

- *Tit for Tat with Forgiveness* – After a defection, j chooses to play the agreed strategy with some small probability. Otherwise j behaves like a player implementing tit-for-tat.

The question then arises: which strategy is the best? [Axelrod_84] reports on a study where participants were invited to code programs that would act as a player in a repeated Prisoners' Dilemma with all players facing off pairwise in a round-robin tournament. The best deterministic strategy entered into the tournament was tit-for-tat with the best overall strategy being a form of tit-for-tat with forgiveness when imperfect signaling was introduced (addressed in Section 4.6.3). This led Axelrod to suppose that successful punishment and reward should be "nice" (initially cooperate), "retaliating" (punish defections), "forgiving" (occasionally autonomously revert back to cooperating), and "non-envious" (not trying to outscore its opponent).

Some twenty years later, this tournament was run again, but this time a different strategy won, or more accurately a different *team* of strategies won [Grossman_04]. In this tournament, researchers could submit multiple programs and one school submitted programs coded so that each program would play a uniquely identifiable sequence of actions to start the game. If two programs from the same team played each other, then one program would switch to always confess while the other switched to always deny – thereby maximizing the payoff of one program and minimizing the payoff of the other. When one of these programs recognized that its opponent was not on its team, it would immediately switch to always confess which minimizes its opponent's payoff.[10] In the end the collaborating programs ended up with the top three scores and most of the bottom scores. So as a somewhat intuitive result, being nice is a good strategy, but gaming the system is even better.

[10] A similar strategy can be employed in winner-take-all poker wherein a team of players collaborate to give their stacks to one particular player. With the "big stack" this player normally has enough of an advantage to win the entire pot. As would be expected, this strategy is frowned upon in reputable circles.

Frequently, players may want to enforce different equilibria. For example, consider the game illustrated in matrix form in Figure 4.19. Given the choice, the row player would prefer to enforce (*b*,*B*) as an equilibrium and the column player would prefer to enforce (*a*,*B*). If these players attempt to punish to enforce either action tuple as an equilibrium without agreeing first coming to some agreement, then play could be very messy with round after round of punishment before one player gives in.

Γ	*A*	*B*
a	(0,0)	(5,10)
b	(0,0)	(10,5)

Figure 4.19: A Game Where Players Would Desire to Enforce Different Equilibria

Thus while not generally discussed in the context of repeated games, when players wish to enforce different equilibria, the game changes from a traditional game to a bargaining game – a topic beyond the scope of this text.[11] Due to the numerous different objectives and device capabilities likely to be encountered, it seems probable that cognitive radios will frequently be operating in networks where they would wish to enforce different equilibria. Thus if cognitive radios are to employ punishment algorithms, the cognitive radios should include some mechanism for negotiation so they can settle upon a common strategy to enforce.

4.4 Desirability[12]

The most typically encountered criteria in the game theory literature that demonstrates that a NE is desirable is *Pareto optimality* also known as *Pareto efficiency*, e.g., [Sung_03], [Krishnaswamy_02], and [Hayajneh_04], defined formally in Definition 4.9.

> **Definition 4.9**: *Pareto optimality*
> An action vector, a^*, is said to be *Pareto optimal* if there exists no other action vector, $a \in A$ such that $u_i(a) \geq u_i(a^*)\ \forall i \in N$ and $u_j(a) > u_j(a^*)$ for at least one player $j \in N$.

[11] However, the textbook version of this dissertation will consider bargaining games as negotiation is a key cognitive radio functionality.
[12] This section is largely a continuation of a debate the author started at his preliminary examination when he declared Pareto optimality to be an "almost worthless" concept for cognitive radio analysis. The casual reader may wish to skip the material presented in this section after Figure 4.20 as material subsequent to Figure 4.20 is an extended and jargon laden justification of the author's opinion of the utility of Pareto optimality in cognitive radio analysis.

Consider the Cognitive Radios' Dilemma whose game matrix is reproduced in Figure 4.20. This game has three Pareto optimal outcomes – (n,N), (w,N), and (n,W) – and only the NE is not Pareto optimal. This highlights a key implication for NE and optimality – an NE is not necessarily optimal nor are optimal points necessarily NE.

Γ	N	W
n	(9.6,9.6)	(3.2,21)
w	(21,3.2)	(7,7)

Figure 4.20: The Cognitive Radios' Dilemma has Three Pareto Optimal Action Vectors.

This author holds the controversial opinion that Pareto optimality is not particularly useful as a tool to aid cognitive radio algorithm design, especially in light of the assumption that the network designer will always have some specific design function, $J: A \rightarrow \mathbb{R}$ (as proposed in Chapter 3), in mind. Specifically, the following are significant weaknesses in the Pareto optimality concept when applied to the design and analysis of cognitive radio algorithms as opposed to simply solving for the maximizers of J.

(1) *Imprecision* – The large number of Pareto optimal states limits its predictive power. For instance, in the cognitive radios' dilemma (n, N) would generally be considered the optimal point for the cognitive radios' dilemma, but (n, N) is just one of three Pareto optimal points. In zero-sum games, such as paper-rock-scissors, *every* action tuple is Pareto optimal. In general, the Pareto front is quite large [Rondeau_04], but most of these will reasonably not be maximizers of J.

(2) *Cost* – Frequently, though not always, the optimal point for J will also be a Pareto optimal point. However, even in the case where Pareto optimality can narrow the search space for J, this is an inefficient approach as determining Pareto optimality can require on the order of $|A|^2|N|$ calculations. In contrast, solving for the action tuple that maximizes the sum of utilities requires on the order of $|A||N|$ calculations.[13] Particularly in light of the

[13] Each comparison of action tuples for Pareto optimality requires $|N|$ comparisons. To determine if an action tuple is Pareto optimal then requires |A| assessments of comparative Pareto optimality. To find all Pareto optimal points, this process must be repeated |A|-1 times. In theory this could be reduced, but would still have the total computations on the order of $|A|^2|N|$. To calculate the action tuple that maximizes the sum of utilities requires |N| additions for the sum utility of an action tuple. Then |A| comparisons and at most |A|

fact that typically |A|>>|N|, evaluating Pareto optimality will frequently be more costly than solving for the maximizers of *J*.

(3) *Incompleteness* – There exist occasions where Pareto dominated solutions are desirable. For instance suppose a cognitive radio designer has available cognitive algorithms that yield the cognitive radios' dilemma. Because of the existence of a legacy system which the radios cannot detect, the FCC has mandated a policy that when two or more cognitive radios are present, the system must operate as an underlay system which [Menon_06] has indicated yields superior legacy protection when compared to a cognitive overlay system. Such a concern for undetectable legacy systems is reflected in 802.22 (TV receivers) and 802.11h (satellites) – the two most prominent cognitive radio standards. This could be resolved by including the legacy systems into the game, but perverts the definition of a game as the game now includes players that cannot make choices.

(4) *Counter Indicators* – There exist points which are Pareto optimal and are undesirable. Example 4.12 gives an example of such an undesirable Pareto optimal point for a power control algorithm. More generally, there may be considerations external to the cognitive radios (e.g., legacy radios, implementation complexity, or security concerns) which make operating points that maximize the sum of cognitive radio utilities (a utilitarian Pareto optimal point) quite undesirable.

Example 4.12: SINR maximizing power control

In [Neel_04a] we briefly considered a single cluster DSSS (Direct Sequence Spread Spectrum) network with a centralized receiver where all of the radios are running power control algorithms in an attempt to maximize their signals' SINR at the receiver.

A normal form game, $\Gamma = \langle N, A, \{u_i\} \rangle$, for this network can formed with the cognitive radios as the players, the available power levels as the action sets and the utility functions given by (4.21) where *K* is the statistical cross-correlation of the signals.

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store operations (a total of 2|A|) must be used to find the maximum(s) giving a total number of operations on the order of |*A*||*N*|.

$$u_i(\mathbf{p}) = h_i p_i \Big/ \left((1/K) \sum_{k \in N \setminus i} h_k p_k + s \right) \qquad (4.21)$$

As might be expected, the unique NE for this game is the power vector where all radios transmit at maximum power. This outcome can be verified to be Pareto optimal as any more equitable power allocation will reduce the utility of the radio closest to the receiver, and any less equitable allocation will reduce the utility of the disadvantaged nodes.

However, this is not a network we would want to implement because of the following:

(1) This state greatly reduces capacity from its potential maximum due to near-far problems (unless our network is in the unlikely configuration of having all radios the same radius from the receiver).

(2) The resulting SINRs are unfairly distributed (the closest node will have a far superior SINR to the furthest node).

(3) Battery life is greatly shortened.

Other than the abundance of Pareto optimal points (part of the imprecision issue) and its relative cost, the weaknesses of applying the Pareto optimality concept to cognitive radio networks when the designer wishes to maximize J derives from the fact that J and the Pareto optimality are two independent functions so it is an odd assumption that solving for the maximizers of one function will provide insights into the maximizers of another function. This same phenomenon is the source of common criticism that Pareto optimal points permit decidedly unfair solutions (as was the case in Example 4.12) and the source of Amartya Sen's criticism of Pareto optimality that Pareto optimality is incompatible with Liberalism[14] [Sen_70]. Whether fairness, Liberalism, or system capacity, there are many goals a designer may have other than Pareto optimality.

[14] Sen defined Liberalism as permitting each player in a game (or society) as permitting each individual to have control over at least one aspect of an outcome, particularly when the outcome only directly relates to that player. As an examp le of Liberalism Sen cited permitting a person to paint their walls pink instead of white with all other aspects of the outcome (society) being the same for the person and society.

With these weaknesses in mind, one might ask why is Pareto efficiency so popular? In short, it is an excellent tool when analyzing *human* evaluated outcomes. Most people do not make choices based on the cardinal relationships of utility functions; instead they make decisions based on ordinally related preferences. This ordinality makes it impossible to perform interpersonal utility comparisons over varying outcomes which is necessary for evaluating utilitarianism and some measures of fairness and all social welfare functions that attempt to assign cardinal relationships to an ordinal phenomenon. Further difficulties in applying social welfare functions to the study of the interactions of people include their implicit paternalism (e.g., "You may think you prefer X, but in the long run you'll thank me for making you take Y.") which many people reject and the difficulty in coming to a consensus as to which function to apply – whether fairness or utilitarianism is the best welfare function remains a serious topic of debate in socio-economic to this day.

Now reconsider the definition of Pareto optimality. Earlier, Pareto optimality was defined in terms of cardinal utility functions in Definition 4.9, but in reality Pareto optimality is intended as a measure of the ordinal preferences of the players in a game. Specifically, an outcome, o^1, is Pareto optimal if there is no other outcome, o^2, such that o^2 is strictly preferred to o^1 by some player with all other players preferring (strict preference or indifferent) to o^2. With an inability to justify the application of any cardinal relationship or a cardinal social welfare function, Pareto optimality appears to be the most agreeable criteria for saying an outcome is desirable – it is impossible to find any other outcome which the players would prefer with at least one player expressing strict preference.

So Pareto optimality eschews the problem of a lack of cardinal preferences by identifying outcomes which are optimal in an ordinal sense and the problem of interpersonal utility comparisons by treating all players' preferences as equally valid. This leads to the last criticism of Pareto optimality when applied to cognitive radios.

(5) *Cardinality of cognitive radios' preferences* – In all considered cognitive radio designs to date, the radios' preferences are cardinal, not ordinal. Specifically, to date all

radios termed as cognitive radios have evaluated outcomes according to some explicit objective function. So for cognitive radios, it is possible to make interpersonal utility comparisons, it is possible to measure how utilitarian an outcome is, it is possible to quantify how much "happier" or "unhappier" a radio will be if the outcome is changed. So the advantages of Pareto optimality over other optimization approaches when evaluating human outcomes do not carry over to the analysis of cognitive radio networks and only Pareto optimality's disadvantages remain.

So in summary, Pareto optimality is an excellent tool for analyzing the optimality of human interactions as humans employ ordinal relationships. However, due to the comparative drawbacks of Pareto optimality, the fact that cognitive radios have cardinally ordered preferences, and the assumption that all cognitive radio designer will have some pre-existing design objective in mind, when designing a cognitive radio algorithm, it is preferable to adopt the approach of Chapter 3 wherein steady states are evaluated via some network cost function, such as Erlang capacity, that is appropriate to the cognitive radio engineer's (and hopefully the user's) design objectives. However, for those rare occasion where a cognitive radio analyst does not have a clear objective in mind, Pareto optimality may be appropriate, though the author would still argue that a utilitarian measure should be employed as while a utilitarian optimal point is necessarily a Pareto optimal point, it reduces the problem of imprecision.

Following this recommendation, consider again the Cognitive Radios' dilemma. If we seek to achieve a fair outcome, then either (n, N) or (w, w) would be optimal. If we seek to achieve the fair outcome with the highest possible net throughput, then (n, N) would be optimal. If we sought to maximize total throughput, then (n, W) and (w, N) would be optimal. In light of this last result, if the cognitive radios' are permitted to employ multi-stage punishment and reward strategies, then many different mixtures of (n, W) and (w, N) could be enforced as an equilibrium for the game and would be optimal in terms of net throughput and, if equally mixed, fairness.

4.5 Convergence

It makes little sense to speak of convergence of a normal form game as it is defined as having only a single iteration. It also makes little sense to speak of the convergence of a repeated game to an enforceable equilibrium as play is assumed to start and remain at the enforced equilibrium. Accordingly, convergence is more frequently discussed in the context of myopic repeated games and the decision rules that guide the radios' reactions to their observations of the network state.

The remainder of this section is organized as follows. Section 4.5.1 defines the classes of decision dynamics to be studied for convergence. Section 4.5.2 defines important normal form game properties which will enable us to analyze myopic decision rules for convergence. Using the concepts defined in the previous two sections, Section 4.5.3 establishes convergence conditions for myopic repeated games. Section 4.5.4 summarizes these results and uses these results to draw insights into the design of cognitive radio.

4.5.1 Classes of Decision Dynamics

While studying the set of all myopic cognitive radio interaction problems implies an infinite number of decision timing patterns and an infinite number of unique decision rules, it is believed that all currently known autonomously rational myopic cognitive radio dynamics can be categorized into the classes of decision timing and decision rules presented in this section.

This section considers the convergence of 16 classes of cognitive radio dynamics defined by associating one of four classes of decision rule with one of four different classes of decision timings. These combinations are listed in Table 4.3 where the decision rules and decision timing classes are described in Section 4.5.1.1 and Section 4.5.1.2, respectively.

Table 4.3 Considered Classes of Dynamics

	Timings			
Decision Rules	Round-Robin	Random	Synchronous	Asynchronous
Deterministic Best Response				
Exhaustive Better Response				
Random Better Response(a)				
Random Better Response(b)				

(a) and (b) represent denote two different classes of algorithms that are defined in the following sections.

4.5.1.1 Decision Timing Classes

Chapter 2 defined the following classes of decision timings: synchronous decision processes, round-robin decision processes, random decision processes, and asynchronous decision processes. There may exist, however, decision timings not captured by these four classes. For example, some players might never get a chance to adapt, or to only adapt only a finite number of times – two scenarios which are dramatically different from normal expectations and excluded from this analysis. So the following analyzes a very broad class of decision timings, but not an exhaustive set of all possible decision timings.

4.5.1.2 Decision Rule Classes

Continuing the notation from Chapter 3, a decision rule is a mapping $d_i : A \to A_i$ that defines the action that a cognitive radio based on an observation of the network state. In general, this work constrains itself to autonomously rational decision rules as defined in Definition 4.3.

While an infinite number of unique decision rules are possible and cognitive radios generally implement very specific decision rules, all autonomously rational myopic decision rules implemented by cognitive radios can be placed into the following classes.

Definition 4.10: *Best Response Dynamic*
A decision rule $d_i : A \to A_i$ is a best response dynamic if each adaptation would maximize the radio's utility if all other radios continued to implement the same waveforms, i.e., $d_i(a) \in \{b_i \in A_i : u_i(b_i, a_{-i}) \geq u_i(a_i, a_{-i}) \forall a_i \in A_i\}$

For instance, radios implementing a power control algorithm according to the standard interference function of Chapter 3 with a utility given by the negation of the distance between observed SINR and target SINR would be implementing a best response dynamic.

Definition 4.11: *Better Response Dynamic*
A decision rule $d_i : A \to A_i$ is a better response dynamic if each adaptation would improve the radio's utility if all other radios continued to implement the same waveforms, i.e., $d_i(a) \in \{b_i \in A_i : u_i(b_i, a_{-i}) > u_i(a_i, a_{-i})\}$.

If a radio always adapts when a better response exists, we say that the better response is *exhaustive*. If the radio is following a deterministic rule to choose which adaptation it makes out of all possible better responses, we say that the better response decision rule is a *deterministic*. For example a cognitive radio implementing a gradient search algorithm is a implementing a deterministic better response and a best response dynamic where the best response is always a single-valued function is a deterministic better response. Note that a deterministic better response dynamic need not be an exhaustive better response dynamic. For instance, a cognitive radio implementing a gradient search may become trapped at a local maximum when other actions would yield better performance.

One class of algorithms not subject to this limitation are *random better response dynamics*.

Definition 4.12: *Random Better Response Dynamic* (*)
A decision rule $d_i : A \to A_i$ is a random better response dynamic if for each $t_i \in T_i$, radio i chooses an action from A_i with nonzero probability and implements the action if it would improve its utility.

Random better response dynamics where all actions in A_i have the same nonzero probability of being chosen are termed *uniform*; if this condition does not hold, the dynamics are termed *nonuniform*. Whether considering a uniform or nonuniform random better response dynamic, it is assumed that all actions have a nonzero probability of being chosen. Note that it is also possible to have a *random* or *deterministic best response dynamic* when multiple best responses are available.

[Friedman_01] considers a class of algorithms very similar to a uniform random better response dynamic, but instead eliminates the player's current action from A_i when randomly selecting an action.

Definition 4.13: *Friedman's Random Better Response* [Friedman_01]
Player i chooses an action from $A_i \backslash b_i$ where b_i is player i's current action according to a uniform random distribution. If the chosen action would improve the utility of player i, it is implemented, otherwise, the player continues to play b_i.

[Friedman_01] specifically attaches this decision rule to a random decision timing pattern. For some situations, these distinctions are inconsequential. However, the broader treatment of decision timings and the inclusion of the current action as part of the random decision process permits us to more accurately model the behavior of cognitive radios implementing genetic algorithms as is done in [Rondeau_04]. In subsequent discussions of the convergence properties of random better response dynamics, we highlight situations where this distinction impacts convergence.

4.5.2 Stage Game Properties

The convergence properties of myopic decision dynamics are largely defined by the properties of their stage games, which is a normal form game. So to analyze the convergence of myopic decision dynamics, we turn to the following normal form game properties: IESDS solvable games, the *finite improvement property* (FIP), and the *weak finite improvement property* (weak FIP). Throughout this discussion, we assume a finite normal formal game defined by the tuple $\Gamma = \langle N, A, \{u_i\} \rangle$.

While IESDS solvable games were defined in section 4.3.1.1, the remaining terms are defined in the following sections. While this section is only considering finite games, this should not be taken to mean that only finite games are IESDS solvable or only finite games have weak FIP. For instance, the SINR maximizing cognitive radio network in Example 4.12 is IESDS solvable (trivially so) as transmitting at maximum power is a strictly dominant strategy.

4.5.2.1 Improvement Path Terminology

Before defining FIP and weak FIP, a number of terms need to be introduced.

Definition 4.14: *Path* [Monderer_96]
A path in Γ is a sequence $g = (a^0, a^1, \ldots)$ such that for every $k \geq 1$ there exists a unique player such that the strategy combinations (a^{k-1}, a^k) differs in exactly one coordinate.

Equivalently, a path is a sequence of unilateral deviations. When discussing paths, we make use of the following conventions.

- Each element of g is called a *step*.
- a^0 is referred to as the *initial* or *starting point* of g.
- Assuming g is finite with m steps, a^m is called the *terminal point* or *ending point* of g and say that g has *length* m.

Formally, we a path, $\gamma = \{a^0, a^1, \ldots, a^k, \ldots\}$, is *finite* if there exists an $m \in \mathbb{N}$ such that there is a bijection between γ and a set of the form $\{0,1,2,\ldots, m\}$. Note that this m is the same as the length of γ. γ is *infinite* if there exists no such m.

Definition 4.15: *Improvement Path* [Monderer_96]
An *improvement path* is a path such that for all $k \geq 1$, $u_i\left(a^k\right) > u_i\left(a^{k-1}\right)$ where i is the unique deviator at step k.

The improvement path is a critical concept to understanding the behavior of cognitive radios as all sequences of adaptations formed by autonomously rational myopic decision processes with random or round-robin timing must trace out an improvement path. So by studying a game's improvement paths, we can identify the possible ways that autonomously rational myopic adaptations will move through the state space. For convergence analysis, one of the most important improvement path properties for a game is the Finite Improvement Property (FIP).

Definition 4.16: *Finite Improvement Property* (*FIP*)
A game, $\Gamma = \langle N, A, \{u_i\} \rangle$, is said to have to have the *finite improvement property* if all improvement paths in Γ are finite.

For finite games (finite player set, finite action space), an equivalent formulation of FIP is a game that lacks improvement cycles, a term that defined in the following.

Definition 4.17: *Cycle*
A (closed) *cycle* is a finite path $g = (a^0, a^1,\ldots,a^m)$ where $a^m = a^0$.

A cycle is said to be an *improvement cycle* if it is also an improvement path. The *length* of a cycle g is the number of unique elements in γ. A cycle is said to be *simple* and *closed* if the only repeated elements are the initial point and the terminal point.

Theorem 4.6: Improvement Cycles and FIP
A finite normal form game, $\Gamma = \langle N, A, \{u_i\} \rangle$ has FIP if and only if Γ lacks improvement cycles.
Proof:
($\Rightarrow$) Suppose that a finite game lacks improvement cycles. Then beginning from every initial point, any improvement path cannot contain more elements than $|A|$ (for finite A). To create an improvement path longer than $|A|$, some element of $|A|$ must be repeated in the improvement path. However, this repetition constitutes an improvement cycle.
($\Leftarrow$) Given an improvement cycle γ, an infinite improvement path can be found as $g^{\infty} = (g, g, \ldots)$. Thus Γ does not have FIP.

Another way to demonstrate that a game has FIP is to simply list all of its improvement paths and verify that they are finite. We follow this approach in the following example.

Example 4.13: Improvement Paths in the Prisoners' Dilemma

Consider the 2x2 game shown in matrix representation in Figure 4.21. A complete listing of the improvement paths for this game is given in Table 4.4. Note that for notational convenience, compound improvement paths are listed. For example improvement path γ_5 is listed as $(g_1, (b, B))$ to denote the sequence $(a, A), (a, B), (b, B)$ where $g_1 = (a, A), (a, B)$. From our exhaustive listing, we can readily establish that this game has FIP and the longest path has a length of 3. Alternately, we can note that all possible simple and closed cycles are not improvement paths.

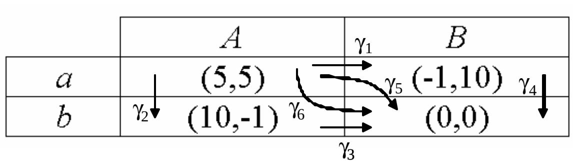

Figure 4.21 Prisoners' Dilemma Game Matrix for Improvement Path Analysis

Table 4.4 Improvement Paths for Game Presented in Figure 4.21

$g_1 = ((a, A), (a, B))$	$g_3 = ((b, A), (b, B))$	$g_5 = (g_1, (b, B))$
$g_2 = ((a, A), (b, A))$	$g_4 = ((a, B), (b, B))$	$g_6 = (g_1, (b, B))$

The other improvement path property that needs to be introduced is the *weak finite improvement path property* given in Definition 4.18. One important way that a game with weak FIP may differ from a game with FIP is that a game with weak FIP may have improvement cycles as we show in Example 4.14. An example of a normal form game that has neither FIP nor weak FIP is shown in the Paper-Rock-Scissors game of Example 4.1.

Definition 4.18: *Weak Finite Improvement Property* (*weak FIP*)
A game, $\Gamma = \langle N, A, \{u_i\} \rangle$, is said to have to have the *weak finite improvement property* if for all $a \in A$ there exists a finite improvement path that terminates in an NE.

Example 4.14 A Game with Weak FIP

An example of a game with weak FIP but not FIP is shown in Figure 4.22. Here an improvement cycle exists - (a,A), (a,B), (b,B), (b,A),(a,A) - and a NE also exists - (c,C) – but starting from any action vector, it is possible to find a finite improvement path that terminates in the NE (c, C). Also note that that this game is IESDS solvable as the bottom row strictly dominates all other actions for the row player and the rightmost column strictly dominates all other actions for the column player.

Γ	*A*	*B*	*C*
a	(1,-1)	(-1,1)	(0,2)
b	(-1,1)	(1,-1)	(1,2)
c	(2,0)	(2,1)	(2,2)

Figure 4.22: A game with weak FIP. (Taken from Figure 2 in [Neel_04a]) The game has an improvement cycle (shown in the arrow) and a NE (circled).

4.5.2.2 General FIP and Weak FIP Properties

In addition to the convergence properties discussed in Section 4.5.3, FIP and weak FIP have several valuable properties. By definition, all games with weak FIP have an NE, but all games with FIP also have an NE.

Theorem 4.7: *FIP and NE existence*
All games with FIP have at least one Nash equilibrium.
Proof: Given a game with FIP, there must be at least one action tuple, a^*, from which there exists no profitable unilateral deviation (otherwise the game would not have FIP). This action tuple a^* must be a Nash equilibrium as there exists no other $a \in A$ such that $u_i\left(a_i, a_{-i}^*\right) > u_i\left(a^*\right)$.

As might be implied by the terms, all games with FIP also have weak FIP.

Theorem 4.8: *FIP and weak FIP*
If $\Gamma = \left\langle N, A, \{u_i\}\right\rangle$ has FIP and is finite, then Γ has weak FIP.

Proof: Suppose Γ has FIP but not weak FIP. Then there exists some $a \in A$ for which there is no improvement path that terminates in an NE. For finite A, this means that a must be a step in an improvement cycle. Yet by Theorem 4.6 a game with FIP cannot have an improvement cycle. Thus for all $a \in A$ there must be an improvement path that terminates in an NE so Γ has weak FIP.

[Friedman_01] provides the following list of games in addition to FIP games that have weak FIP:

- IESDS solvable games,
- Quasi-acyclic games,
- Finite supermodular games, and
- Continuous, two-player, quasi-concave games.

As noted for IESDS solvable games, a round robin best response dynamic is everywhere convergent to the game's unique NE, thus supplying the requisite improvement path for weak FIP. As defined in [Friedman_01], a *quasi-acyclic game* is a normal form game such that from any $a \in A$ there exists a finite sequence of strict best responses that terminate in an NE (thus supplying the requisite improvement path in the definition).

Supermodular games merit a longer discussion and are addressed in Chapter 8.

In a lengthy proof, [Friedman_01] shows that weak FIP holds for normal form games with the following properties which [Friedman_01] terms *generic, two-player, quasi-concave games*.[15]

- $N = \{1,2\}$
- $\forall i \in N$, $A_i \subset \mathbb{R}$ and is compact and convex.
- $\forall i \in N$, u_i is strictly quasi-concave for all $a_i \in A_i$ and is twice continuously differentiable.

Differentiating these games from quasi-acyclic games, [Friedman_01] provides the following example.

Example 4.15 Friedman's Generic Two-Player Quasi-Concave Game

Consider a normal form game defined by the following components: $N = \{1,2\}$, $A_i = [0,2]$, $u_1(a) = 4 + (4 - 2a_2)a_1 - a_1^2$, and $u_2(a) = 4 + 2a_1a_2 - a_2^2$. This can be readily verified as one of Friedman's generic, two-player, quasi-concave games. However, it is not a quasi-acyclic game by considering the best response functions given by the following: $B_1(a) = 2 - a_2$ and $B_2(a) = a_1$.

Simultaneous solution of these equations yields (1,1) as an NE. However, starting from (0,0) and applying a round-robin best response dynamic yields the following improvement cycle: (0, 0), (2, 0), (2, 2), (0, 2), (0, 0). Thus this game is not quasi-acyclic. However, we can verify that this starting point is not a counter-example to the game having weak FIP as the following improvement path terminates in an NE in a finite number of steps: (0, 0), (1, 0), (1, 1).

Interestingly, it should be noted that it is not necessary for a game to be finite for it to have FIP. Consider the two player game where $A_1 = A_2 = [0,1]$ and the utility functions

[15] Many readers may find this term misleading as the game's requisite conditions are quite specific, i.e., not generic, and strict quasi-concavity of utility functions as opposed to quasi-concavity is required.

for this game be given by (4.22). As there is no improvement path in this game longer than two steps, the game has FIP.[16]

$$u_i(a) = \begin{cases} 1 & a_i = 0 \\ 0 & otherwise \end{cases} \quad (4.22)$$

Elided in this discussion is how to identify games that have FIP beyond an exhaustive listing of improvement paths and cycles. In Chapter 5, we consider *potential games* and *generalized ordinal potential games* which are coincident with the class of games with FIP.

4.5.3 Convergence Properties

To study convergence properties, the myopic repeated game defined by the tuple $\Gamma = \langle N, A, \{u_i\}, \{d_i\}, T \rangle$ can be broken into its constituent normal form game, $\Gamma = \langle N, A, \{u_i\} \rangle$, the decision rules $\{d_i\}$, and the decision timings T. To facilitate the establishment of convergence and convergence rates, we rely heavily on the absorbing Markov chain theory introduced in Chapter 3.

4.5.3.1 Convergence of IESDS Solvable Games

An IESDS solvable game is characterized by a single action tuple remaining after the synchronous or round-robin elimination of strictly dominated strategies. This has important consequences for the convergence of best response algorithms.

Best Response Decision Rules

Intuitively, best response decision rules converge for IESDS solvable games. Putting this intuition on a firmer foundation, consider a round of play in any order (and possibly with some players adapting multiple times) such that all *n* players have had at least one chance to play their best responses. As all players have played best responses, no player would have played a strictly dominated action. Because the game is IESDS solvable, at least one player in this round must now have at least one action that it will never return to otherwise the round started at the unique NE or Algorithm 4.2 has failed and the game is

[16] It is believed that this is the first example of an infinite game with FIP.

not IESDS solvable. After the next round of best responses played on this effectively smaller action space (the player that "eliminated" a strictly dominated action will not play that action, reducing the effective action space), some player must eliminate another action from the effective action space. This process continues until no more actions are effectively eliminated, which for IESDS solvable games implies the game has converged to the unique NE.

So assuming best responses, the best response dynamic converges for round-robin, random, synchronous, or asynchronous decision processes.

Example 4.16 Convergence of SINR Maximizing Power Control

Consider the SINR maximizing power control game of Example 4.12 which is IESDS solvable. For simplicity, assume there are three radios, each of which have two power levels available – low, and high which we label as L, and H, respectively. For best response decision rules and random or asynchronous decision timings, this corresponds to an eight state absorbing Markov chain. To determine the expected convergence time, we write the transition matrix for this network under random and asynchronous timings as shown in Table 4.5 and Table 4.6, respectively, assuming equal probabilities for drawing elements from N and $2^N \setminus \varnothing$, respectively.

Table 4.5 Transition Matrix for Random Timing

Q	(L,L,L)	(L,L,H)	(L,H,L)	(L,H,H)	(H,L,L)	(H,L,H)	(H,H,L)	(H,H,H)
(L,L,L)	0	1/3	1/3	0	1/3	0	0	0
(L,L,H)	0	1/3	0	1/3	0	1/3	0	0
(L,H,L)	0	0	1/3	1/3	0	0	1/3	0
(L,H,H)	0	0	0	2/3	0	0	0	1/3
(H,L,L)	0	0	0	0	1/3	1/3	1/3	0
(H,L,H)	0	0	0	0	0	2/3	0	1/3
(H,H,L)	0	0	0	0	0	0	2/3	1/3
(H,H,H)	0	0	0	0	0	0	0	1

R

Table 4.6 Transition Matrix for Asynchronous Timing

Q	(L,L,L)	(L,L,H)	(L,H,L)	(L,H,H)	(H,L,L)	(H,L,H)	(H,H,L)	(H,H,H)
(L,L,L)	0	1/7	1/7	1/7	1/7	1/7	1/7	1/7
(L,L,H)	0	1/7	0	2/7	0	2/7	0	2/7
(L,H,L)	0	0	1/7	2/7	0	0	2/7	2/7
(L,H,H)	0	0	0	3/7	0	0	0	4/7
(H,L,L)	0	0	0	0	1/7	2/7	2/7	2/7
(H,L,H)	0	0	0	0	0	3/7	0	4/7
(H,H,L)	0	0	0	0	0	0	3/7	4/7
(H,H,H)	0	0	0	0	0	0	0	1

R

As these matrices are already in canonical form, we can solve for the expected number of iterations for convergence to the absorbing state by evaluating $(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{1}$ where $\mathbf{Q}$ is the submatrix labeled in the tables, **I** is the identity matrix and **1** is a seven element column of ones. Evaluating this expression we find the expected convergence times given in Table 4.7 and

Table 4.8 for random and asynchronous timings, respectively, where starting in (H,H,H) requires 0 iterations for convergence. Note that when a network can be modeled as an IESDS solvable games with best response decision rules, the network will converge faster with asynchronous timings than with random decision timing on average.

Table 4.7 Expected Convergence Times to (H,H,H) for Different Initial States for Random Timing

	(L,L,L)	(L,L,H)	(L,H,L)	(L,H,H)	(H,L,L)	(H,L,H)	(H,H,L)
Iterations	5.5	4.5	4.5	3	4.5	3	3

Table 4.8 Expected Convergence Times to (H,H,H) for Different Initial States for Asynchronous Timing

	(L,L,L)	(L,L,H)	(L,H,L)	(L,H,H)	(H,L,L)	(H,L,H)	(H,H,L)
Iterations	2.75	7/3	7/3	1.75	7/3	1.75	1.75

Better Response Decision Rules

By playing best responses, players are assured of never playing a strictly dominated action, so convergence was assured. However, the same does not hold for better response decision rules.

Consider the game matrix shown in Example 4.14 which is IESDS solvable. Because of the existence of an improvement cycle, this game provides a counter-example to an assertion that the class of deterministic better response algorithms converge in an IESDS solvable game for any of the four classes of decision timings. However, the inability to guarantee convergence does not extend to random better response decision rules. However, proving this in the most general fashion requires the consideration of games with weak FIP.

4.5.3.2 Convergence of Games with Weak FIP

Recall that a game has weak FIP if for all $a \in A$ there exists a finite improvement path that terminates in an NE and that IESDS games, quasi-acyclic games, supermodular games, and games with FIP all have weak FIP. As this is a very general condition, the requisite improvement path is expected to vary by game model.

For a game that is IESDS solvable, weak FIP is verified by any improvement path formed by round-robin application of best responses. For a quasi-acyclic game, there is always some sequence of strict best responses (not necessarily round-robin) that terminates in an NE. For a game with FIP, any round-robin sequence of best responses suffices as does any round-robin sequence of exhaustive better responses on a finite *A*. While all of these games satisfy weak FIP via round-robin best response improvement paths, this does not hold for the game in Example 4.15 which exhibits a best response improvement cycle. Thus the only assumption that can be made about the class of games with weak FIP is that there always exists some finite improvement path that terminates in a NE.

Best Response Decision Rules

As stated in the preceding, Example 4.15 provides a counter example to any assertion that all members of the class of best response decision rules converges. A deeper examination of that game reveals that not the game would still exhibit the improvement cycle (0, 0), (2, 0), (2, 2), (0, 2), (0, 0) if played under any of the four classes of decision timings.

Deterministic Better Response Decision Rules

Again Example 4.15 and its best response improvement cycle provide a counter example where a deterministic better response rule fails to converge for a game with weak FIP.

Random Better Response Decision Rules

[Friedman_01] studies games with weak FIP and random decision timing which are coupled with the decision rule of Definition 4.13. For finite *A*, it is seen that for a game with weak FIP, this decision rule always has a nonzero chance of stumbling upon an improvement path that leads to an NE. As NEs constitute absorbing Markov states (for all players, there is no action in $A_i \backslash b_i$ that would improve utility for any player), play can be modeled as an absorbing Markov chain. Thus convergence to an NE is assured for a random decision timing.

Although not considered by [Friedman_01], this same decision rule would form an absorbing Markov chain for the asynchronous timing rule considered in this Section. Specifically, for asynchronous timings any finite single player ordering has a nonzero chance of occurring, so there must be a nonzero chance of replicating the same single player ordering generated by random timing.

However, the random better response of [Friedman_01] can fail for synchronous or round-robin timing. As a counter example to the assumption that it would converge for synchronous timings, consider the coordination game shown in Figure 4.23. As can be verified from Table 4.9, if the network starts in state (*a*,*B*) or (*b*,*A*), play enters the cycle (*a*,*B*), (*b*,*A*). Note that while this is not an improvement cycle, it is a cycle that results from proper application of the decision rule for synchronous timings.

Γ	*A*	*B*
a	(1,1)	(0,0)
b	(0,0)	(1,1)

Figure 4.23: A Coordination Game which Can Fail to Converge for the Random Better Response of [Friedman_01] for Synchronous Timings.

Table 4.9: The Markov Transition Matrix for the Game in Figure 4.23 with Synchronous Timings and Decision Rule of Definition 4.13

	(*a*,*A*)	(*a*,*B*)	(*b*,*A*)	(*b*,*B*)
(*a*,*A*)	1	0	0	0
(*a*,*B*)	0	0	1	0
(*b*,*A*)	0	1	0	0
(*b*,*B*)	0	0	0	1

Likewise, it is conceivable that a particular order of play is needed to escape a set of recurrent states and that the required order of play may not be the order reflected in the round-robin timing.[17] These timing difficulties disappear, however, when the random decision rule of Definition 4.12 is employed. The only difference between the two rules is that rather than choosing from $A_i \backslash b_i$, each player chooses from A_i. Importantly, this difference preserves the convergence criteria of the [Friedman_01] algorithm.

Theorem 4.9: Better Responses and Markov Chain Implications (*)
Given a particular decision timing, *T*, if implementing the decision rule of Definition 4.13 forms an absorbing Markov Chain, then implementing the decision rule of Definition 4.12 must also implement an absorbing Markov chain.
Proof: As the set of choices for each adaptation in Definition 4.13 is a subset of the choices of Definition 4.12, then whenever any sequence of state transition generated by Definition 4.13 has a nonzero probability, the sequence must also have a nonzero probability of occurring under Definition 4.12. Further, it is clear that a state is an absorbing state for Definition 4.13, it is also an absorbing state for Definition 4.12. Thus if there exists an absorbing Markov chain for Definition 4.13, it must also exist for Definition 4.12.

Theorem 4.9 immediately implies that for random and asynchronous timings, the decision rule of Definition 4.12 forms an absorbing Markov chain and therefore converges. However, the decision rule of Definition 4.12 forms an absorbing Markov chain for round-robin and synchronous decision timings as well. Specifically, as each radio has a nonzero chance of choosing its current action, there is a nonzero probability of generating the sequence of adaptations corresponding to the action tuple's finite improvement path.

[17] IESDS solvable games, however, do not suffer from this limitation as the order of elimination is not important.

For instance, consider again the game shown in Figure 4.23, but now apply the Decision Rule of Definition 4.12 under synchronous timing. This yields the transition matrix shown in Table 4.10 which is readily verified as an absorbing Markov chain. Thus for games with weak FIP, the proposed decision rule converges for all decision timings and not just for random and asynchronous timings.

Table 4.10: The Markov Transition Matrix for the Game in Figure 4.23 with Synchronous Timings and Decision Rule of Definition 4.13.

	(a,A)	(a,B)	(b,A)	(b,B)
(a,A)	1	0	0	0
(a,B)	0.25	0.25	0.25	0.25
(b,A)	0.25	0.25	0.25	0.25
(b,B)	0	0	0	1

4.5.3.3 Convergence of Games with FIP

As defined FIP in Definition 4.16, a normal form game has FIP if all improvement paths in Γ are finite. This implies that the only requirement for convergence is that the radios act in their own interest and that all unilateral adaptations have a nonzero chance of occurring (recall that improvement paths were defined in terms of unilateral deviations).

Best Response Decision Rules

Whether implemented deterministically or randomly, any sequence of adaptations generated by unilateral best responses necessarily results in an improvement path. As FIP guarantees that these paths are finite, we know that round-robin and random decision best response algorithms converge. As we reasoned before, asynchronous timing has a nonzero probability of generating the same sequence of adaptations generated under random timings. Thus a best response decision rule with asynchronous timing forms an absorbing Markov chain for games with FIP.

However, games with FIP need not converge under synchronous timings with best response decision rules. For example, best response decision rules with synchronous timings applied to the game of Figure 4.23, which has FIP, result in the same cycle seen before.

Deterministic Better Response Decision Rules

If we assume exhaustive better response decision rules, then all improvement paths generated by round-robin, random, and asynchronous decision timings must terminate in an NE thereby implying convergence. However, the game of Figure 4.23 again provides a counter-example to synchronous convergence for deterministic better response decision rules.

Random Better Response Decision Rules

As shown previously, all games with FIP have weak FIP. Thus when played on games with FIP, the random better response of Definition 4.12 converges for all four timing classes, and the random better response of Definition 4.13 converges for random and asynchronous decision timings. However, the decision rule of Definition 4.13 also converges when applied to round-robin timing rules. Specifically, this combination generates necessarily generates an improvement path and by FIP, this improvement path must be finite. As all possible better responses have a chance to be played, this algorithm must converge to a NE.

4.5.4 Convergence Summary and Conclusions

The criteria for assured convergence for the classes of games, decision rules, and decision timings are summarized in Table 4.11. Note that the broadest convergence conditions hold under random and asynchronous timings and for the random better response rules. This implies two important results for myopic cognitive radio network design. First, cognitive radio networks should in general be designed so decision timings are randomized instead of synchronized – a good thing as synchronized algorithms generally do not scale as well as randomized algorithms due to the increased overhead inherent in the synchronization process. In general, the decision engines in cognitive radios should support decision rules that satisfy the class of decision rules of Definition 4.12 as that rule supports the broadest range of convergence conditions. However, it should be noted that when any of the other classes of decision rules converge, it should be possible to design algorithms that converge faster than the class of decision rules of Definition 4.12. This design suggestion appears to be reasonable as Virginia Tech has built a cognitive radio whose decision engine implements a decision process in this class of algorithms [Rondeau_04].

Table 4.11: Convergence Criteria

	Timings			
Decision Rules	Round-Robin	Random	Synchronous	Asynchronous
Best Response	1,3	1,3	1	1,3
Exhaustive Better Response	3	3	-	3
Random Better Response[(a)]	1,2,3	1,2,3	1,2,3	1,2,3
Random Better Response[(b)]	1,3	1,2,3	1	1,2,3

(a) Definition 4.12, (b) Definition 4.13, 1. IESDS, 2. Weak FIP, 3. FIP

For all of the surveyed game model properties, it was seen that the choice of best response did not influence convergence criteria. However, Chapter 8 presents a class of games for which different choices of best responses can lead to convergence to different steady-states – an unsurprising result. Likewise, while we only considered uniform random distributions of decision timings and better responses in the examples, it should be noted that the absorbing Markov chain properties used to establish convergence are also ensured for any non-uniform probability distribution that preserves the non-zero probabilities of the uniform distribution or more generally, preserves the non-zero probability that the relevant improvement path occurs. This section also ignored what can be said about convergence criteria for these games when the action space is infinite. The topic of convergence in games with infinite action sets in Chapters 5 and 8.

It might be speculated that the random better response would converge for any finite game with an NE (obviously, a myopic process played on a game without an NE, such as Paper-Rock-Scissors, cannot converge). However, this is not generally the case as evidenced by Figure 4.24 which will remain trapped in the improvement cycle (*a*,*A*), (*a*,*B*), (*b*,*B*), (*b*,*A*), (*a*,*A*).

Γ	*A*	*B*	*C*
a	(1,0)	(0,1)	(-10,-10)
b	(0,1)	(1,0)	(-10,-10)
c	(-10,-10)	(-10,-10)	(1,1)

Figure 4.24 A Game with an NE, but not Weak FIP, FIP, or IESDS.

In fact, if a finite game lacks an improvement path to an NE from an action tuple, then by definition, no sequence of myopic unilateral deviations can converge to an NE from that action tuple.

Theorem 4.10: *Weak FIP and Deterministic Myopic Convergence* (*)
If a finite game lacks the weak FIP property, then there exists at least four action tuples for which all autonomously rational myopic decision processes will not converge under round robin and random timings.
Proof: Note that all rational myopic unilateral decision processes are constrained to follow improvement paths. By definition, lack of weak FIP implies that there must be at least one action tuple for which no improvement path leads to an NE. As this action tuple cannot be an NE (otherwise there exists a trivial improvement path of length 0 that leads to an NE), there must be at least one improvement path that leads away from the action tuple. As this improvement path cannot terminate in an NE, it must cycle through at least four action tuples that also do not have improvement paths to an NE (an improvement path of length four is the shortest possible improvement cycle) and thus any autonomously rational myopic decision processes will not converge under round robin and random timings when starting from these action tuples.

Because rational myopic convergence is not assured without weak FIP, identifying that the intended algorithm satisfies weak FIP should be a step included in the process of designing myopic cognitive radio algorithms.

In games without weak FIP, convergence requires *higher-order rationality* – the ability of players to reason beyond a single stage. For instance, in the game of Figure 4.24, the column player could play C – an irrational play in a myopic repeated game – knowing that the row player would respond with c which then places the network in a desirable steady-state. However, higher order-rationality implies that a cognitive radio minimally know the values of the other radio's utility functions and frequently know the decision rules and internal states of the other radios. An alternate strategy to overcoming this lack of assured convergence would be to permit the radios to implement punishment and reward strategies, though as previously discussed, this generally requires the radios be able to negotiate and may not scale well. As either approach significantly adds to the complexity of a cognitive radio implementation, it seems preferable to seek out cognitive radio algorithms whose goals and actions satisfy weak FIP. Beyond the broad classifications considered in this section, Chapters 5 and 8 present readily applied techniques for identifying when cognitive radio goals and action spaces satisfy weak FIP.

4.6 Impact of Noise

Any practical implementation of a cognitive radio network occurs in a noisy environment which corrupts the observation process. For instance, consider a radio observing the spectral energy across the bands defined by the set C where each radio k is choosing its band of operation f_k. If noise were not present, radio i's observation of the signal energy in channel c_k could be described by (4.23)

$$o_i(c_k) = \sum_{k \in N} g_{ki} p_k q(c_k, f_k) \tag{4.23}$$

where g_{ki} is the gain from radio k to where radio i is taking its measurement, p_k is the transmit power of radio k and $q(c_k, f_k) = 1$ if $f_k = c_k$ and $q(c_k, f_k) = 0$ otherwise.

Because of noise in the receiver, (4.23) will be realized as (4.24)

$$\tilde{o}_i(c_k) = \sum_{k \in N} g_{ki} p_k q(c_k, f_k) + n_i(c_k, t) \tag{4.24}$$

where $n_i(c_k,t)$ is the noise at the receiver in channel k as measured at time t. If each radio is attempting to operate on the channel with minimum interference ($u_i = -o_i$), then instead of adapting to the channel that minimizes o_i, the radio will operate on the channel that minimizes $\tilde{o}_i$. Thus noisy observations lead to corrupted cognitive radio goals which lead to the implementation of actions that the analyst would interpret as being in error.

Continuing to rely the assumption that communications theory permits us to map actions to goals, given cognitive radio network $\Gamma = \langle N, A, \{u_i\}, \{d_i\}, T\rangle$, we can model a *noise corrupted cognitive radio network* by $\Gamma = \langle N, A, \{\tilde{u}_i\}, \{d_i\}, T\rangle$ where $\tilde{u}_i$ is defined as shown in (4.25)

(*Noisy utility*) $$\tilde{u}_i(a,t) = u_i(a) + n_i(a,t) \tag{4.25}$$

where $n_i(a,t)$ is a stochastic process corrupting the evaluation of the player's utility function at time t. So when a rational decision maker believes that $\tilde{u}_i(b_i, a_{-i}, t) > \tilde{u}_i(a_i, a_{-i}, t)$, it may be because b_i is a better choice, i.e.,

$u_i(b_i, a_{-i}) > u_i(a_i, a_{-i})$, or because noise has corrupted the observation at time t. For our purposes, this means that instead of implementing b_i as would have normally been predicted, the radio may implement a_i.

Under the reasonable assumption that $n_i(t)$ is unbounded (perhaps because the noise source is Gaussian) and u_i is bounded, there is a nonzero (though perhaps very small) probability that $\tilde{u}_i(b_i, a_{-i}, t)$ is less than $\tilde{u}_i(a_i, a_{-i}, t)$ regardless of how much greater $u_i(b_i, a_{-i})$ is than $u_i(a_i, a_{-i})$. So under normal operating conditions, a cognitive radio always has a nonzero chance of making a mistake.

Game theoretic analyses typically attribute these decision errors to mistakes in implementation, e.g., a *tremble* in a player's hand as it tries to implement its chosen action. For people, this is a phenomenon we are all familiar with. What basketball player has not missed a shot? Who has not accidentally handed over the wrong amount of cash at a restaurant? Who has not accidentally written a "tpyo"?

In terms of the effect on play, a corrupted observation can appear to be analytically equivalent to a mistake in implementation so analysis can frequently proceed with either an assumption of errors in implementation or observation. However, there are some subtle, though important, analysis differences and in terms of design, that will be highlighted in the following. Also it is important for a cognitive radio designer to be aware that even with the reasonable assumption of perfect implementation of decisions, errors will still occur so a cognitive radio designer should include mechanisms for minimizing the negative impacts of these errors.

4.6.1 Noise and Nash Equilibria

With noise in the system, an NE is no longer an ideal equilibrium for the system as play occasionally leaves the NE. Using the assumption that errors result from implementation mistakes, a different equilibrium definition can be defined – the *trembling hand equilibrium* which is a variation on the mixed strategy NE.

Definition 4.19: *Trembling Hand Equilibrium* ([Osborne_94], Definition 248.1)
A *trembling hand equilibrium,* $\tilde{a}$ of a finite game is a mixed strategy profile $\boldsymbol{a}$ with the property that there exists an infinite sequence (a^k) of mixed strategy profiles that converges to $\tilde{a}$ such that for each player i $\tilde{a}_i$ is a best response to $\tilde{a}_{-i}^k$ $\forall k$.

Intuitively, this definition makes sense as a noisy equilibrium for implementation errors as the trembling hand equilibrium is a best response to the errors (realized as mixed strategies). However, this definition is difficult to apply. A refinement concept is provided in ([Osborne_94], p. 248) – a mixed strategy $\tilde{a}$ is a trembling hand equilibrium only if $\tilde{a}$ is not dominated by some other mixed strategy.

The next section introduces a technique for identifying an equilibrium based on improvement paths and noise which we believe is more suitable to cognitive radio analysis as it permits a more direct calculation of the distribution of states.

4.6.2 Noise and Decision Processes

Instead of evaluating games for trembling hand equilibria, we can also analyze the impact of noise on improvement paths to characterize network behavior. As in Section 4.5, this discussion begins with an introduction of myopic decision rules, but now operating in the presence of noise. [Friedman_01] considers a noisy better response based on a trembling hand.

Definition 4.20: *Friedman's Noisy Random Better Response* [Friedman_01]
Player i chooses an action $a_i \in A_i \backslash b_i$ where b_i is player i's current action according to a uniform random distribution. If $u_i(a_i,a_{-i}) > u_i(b_i,a_{-i})$, then a_i is implemented, however, if $u_i(a_i,a_{-i}) \leq u_i(b_i,a_{-i})$, then player i still switches to a_i with nonzero probability ρ.

[Friedman_01] notes that when all players are implementing the dynamic given in Definition 4.20 with random timing, the game can be modeled as an ergodic Markov chain. By Theorem 3.6, this implies that there exists a unique limiting and stationary distribution for all initial distributions p^0. This means that solving the eigenvector problem given by (4.26) where $\mathbf{P}$ is the transition matrix and an eigenvalue of 1 is assumed yields the equilibrium behavior for the system.

$$p^{*T}\mathbf{P} = p^{*T} \tag{4.26}$$

Similar results can be noted for the following three noisy dynamics defined by substituting $\tilde{u}_i$ for u_i into the best, better, and random better responses.

Definition 4.21: *Noisy Best Response Dynamic* (*)
A decision rule $\tilde{d}_i : A \times T \rightarrow A_i$ is a noisy best response dynamic if each adaptation would maximize the radio's noisy utility if all other radios continued to implement the same waveforms, i.e., $\tilde{d}_i(a) \in \left\{ b_i \in A_i : \tilde{u}_i(b_i, a_{-i}, t) \geq \tilde{u}_i(a_i, a_{-i}, t) \, \forall a_i \in A_i \right\}$

Definition 4.22: *Noisy Better Response Dynamic* (*)
A decision rule $\tilde{d}_i : A \times T \rightarrow A_i$ is a noisy better response dynamic if each adaptation would improve the radio's utility if all other radios continued to implement the same waveforms, i.e., $\tilde{d}_i(a) \in \left\{ b_i \in A_i : \tilde{u}_i(b_i, a_{-i}, t) > \tilde{u}_i(a_i, a_{-i}, t) \right\}$.

Definition 4.23: *Noisy Random Better Response Dynamic* (*)
A decision rule $\tilde{d}_i : A \times T \rightarrow A_i$ is a random better response dynamic if for each $t_i \in T_i$, radio i chooses an action from A_i with nonzero probability and implements the action if it would improve $\tilde{u}_i$.

For all three dynamics, every $a_i \in A_i$ has a nonzero probability of being the next action of player i. Thus for synchronous and asynchronous timings, it is apparent that the transition matrix has no zero entries, implying that the Markov chain is ergodic. For a random timing, it is clear that the Markov chain is irreducible, positive recurrent, and aperiodic implying that the Markov chain is ergodic.

For round robin timing, however, the system is periodic in that once a state is left, the system can only return in multiple of n iterations implying that the system is not ergodic. However, if we define an "iteration" as a complete round, then this Markov chain is aperiodic and the Markov chain is ergodic.

Earlier it was noted that mistakes in observation can lead to effects similar to mistakes in implementation. In this case, the three proposed decision rules with noisy observations and the trembling hand implementation of [Friedman_01] all yield ergodic Markov chains. However, as the following example makes clear, the different sources of error result in different steady-state distributions.

Example 4.17: Noisy DFS Decision Processes

Consider a network consisting of three access points implementing DFS over two channels with random locations. Each access point seeks to minimize the interference that it receives from the other two access points as measured at its own location. Assuming equal transmit powers, p, each radio would observe its interference as shown (4.23). Further, let us assume that these access points are located so that the symmetric link gain matrix shown in Table 4.12 results.

Table 4.12: Link Gain Matrix

g_{ik}	1	2	3
1	1	0.5	0.1
2	0.5	1	0.3
3	0.1	0.3	1

Assuming a transmit power of $p = 1$, the possible noiseless observations for this system are shown in Table 4.13.

Table 4.13 Noiseless Observations

(f_1,f_1,f_1)	(f_1,f_1,f_2)	(f_1,f_2,f_1)	(f_1,f_2,f_2)	(f_2,f_1,f_1)	(f_2,f_1,f_2)	(f_2,f_2,f_1)	(f_2,f_2,f_2)	(f_2,f_1,f_1)
(0.6,0.8,0.4)	(0.5,0.5,0.0)	(0.1,0.0,0.1)	(0.0,0.3,0.3)	(0.0,0.3,0.3)	(0.1,0.0,0.1)	(0.5,0.5,0.0)	(0.6,0.8,0.4)	(0.0,0.3,0.3)

One typical DFS algorithm objective, and the one that we use here, is for that each radio is trying to minimize its perceived interference, i.e., $u_i = -o_i$. Assuming the radios are randomly making implementation errors with a probability of 0.1 and adapting according to a random timing scheme, the transition matrix for this system is given by Table 4.14. Note that the two NE for this system – (f_1,f_2,f_1) and (f_2,f_1,f_2) – have the largest probabilities of remaining in the same state.

Table 4.14 Transition Matrix for Better Response, Trembling Hand, r=0.1, and Random Timing

P	(f_1,f_1,f_1)	(f_1,f_1,f_2)	(f_1,f_2,f_1)	(f_1,f_2,f_2)	(f_2,f_1,f_1)	(f_2,f_1,f_2)	(f_2,f_2,f_1)	(f_2,f_2,f_2)
(f_1,f_1,f_1)	0	1/3	1/3	0	1/3	0	0	0
(f_1,f_1,f_2)	1/30	3/10	0	1/3	0	1/3	0	0
(f_1,f_2,f_1)	1/30	0	9/10	1/30	0	0	1/30	0
(f_1,f_2,f_2)	0	1/30	1/3	3/5	0	0	0	1/30
(f_2,f_1,f_1)	1/30	0	0	0	3/5	1/3	1/30	0
(f_2,f_1,f_2)	0	1/30	0	0	1/30	9/10	0	1/30
(f_2,f_2,f_1)	0	0	1/3	0	1/3	0	3/10	1/30
(f_2,f_2,f_2)	0	0	0	1/3	0	1/3	1/3	0

Solving the eigenvalue problem $p^{*T}\mathbf{P} = p^{*T}$ yields the following vector of steady-state probabilities. Note the two NE for this system – (f_1,f_2,f_1) and (f_2,f_1,f_2) – have the largest probabilities of remaining in the same state.

(f_1,f_1,f_1)	(f_1,f_1,f_2)	(f_1,f_2,f_1)	(f_1,f_2,f_2)	(f_2,f_1,f_1)	(f_2,f_1,f_2)	(f_2,f_2,f_1)	(f_2,f_2,f_2)
0.0161	0.0293	0.3846	0.0699	0.0699	0.3846	0.0293	0.0161

If however, we assume that errors are the result of a noisy observation process, a different distribution results. Specifically, now a best response adapts whenever it believes $\tilde{u}_i(b_i, a_{-i}, t) > \tilde{u}_i(a_i, a_{-i}, t)$. Let us assume that n_i is a zero mean Gaussian stochastic variable with standard deviation of 1. Then the probability that radio 3 will evaluate f_k as better than f_1 from action vector (f_1,f_1,f_1) is given by (4.27)

$$P\left(\tilde{u}_3(f_1, f_1, f_2) > \tilde{u}_3(f_1, f_1, f_2)\right) = \int_0^{\infty} \frac{1}{2\sqrt{p}} e^{-(x-(0+0.4))^2/(4)} dx \tag{4.27}$$

or $Q\left(\frac{-0.4}{\sqrt{2}}\right)$. Generalizing to other possible adaptations for this system, the probability that player i adapts from a_i to b_i is given by (4.28).

$$Q\left(\frac{u_i(b_i, a_{-i}) - u_i(a_i, a_{-i})}{\sqrt{2}}\right) \tag{4.28}$$

With this formula in hand, we can write the transition matrix for the noisy best response as shown in Table 4.15. Note that the two NE for this system – (f_1,f_2,f_1) and (f_2,f_1,f_2) – have the largest probabilities of remaining in the same state, although the probability of making an error is much higher and some transition probabilities are no longer zero, e.g., $(f_1,f_1,f_1) \rightarrow (f_1,f_1,f_1)$.

Table 4.15: Transition Matrix for Best Response and Random Timing with Observations Corrupted by $\mathcal{N}(0,1)$ Gaussian Noise

P	(f_1,f_1,f_1)	(f_1,f_1,f_2)	(f_1,f_2,f_1)	(f_1,f_2,f_2)	(f_2,f_1,f_1)	(f_2,f_1,f_2)	(f_2,f_2,f_1)	(f_2,f_2,f_2)
(f_1,f_1,f_1)	0.3367	0.2038	0.2381	0	0.2214	0	0	0
(f_1,f_1,f_2)	0.1295	0.4813	0	0.1854	0	0.2038	0	0
(f_1,f_2,f_1)	0.0953	0	0.6273	0.1479	0	0	0.1295	0
(f_1,f_2,f_2)	0	0.1479	0.1854	0.5548	0	0	0	0.1119
(f_2,f_1,f_1)	0.1119	0	0	0	0.5548	0.1854	0.1479	0
(f_2,f_1,f_2)	0	0.1295	0	0	0.1479	0.6273	0	0.0953
(f_2,f_2,f_1)	0	0	0.2038	0	0.1854	0	0.4813	0.1295
(f_2,f_2,f_2)	0	0	0	0.2214	0	0.2381	0.2038	0.3367

Solving the eigenvalue problem $p^{*T}\mathbf{P} = p^{*T}$ yields the following vector of steady-state probabilities. Note the two NE for this system – (f_1,f_2,f_1) and (f_2,f_1,f_2) – have the largest probabilities of remaining in the same state, although this is significantly smaller than for implementation errors.

Table 4.16: Steady State Distributions for a Standard Deviation of 1

(f_1,f_1,f_1)	(f_1,f_1,f_2)	(f_1,f_2,f_1)	(f_1,f_2,f_2)	(f_2,f_1,f_1)	(f_2,f_1,f_2)	(f_2,f_2,f_1)	(f_2,f_2,f_2)
0.0709	0.1120	0.1765	0.1406	0.1406	0.1765	0.1120	0.0709

Intuitively, the distribution of states becomes more clustered about the NE as noise power reduces as shown in Table 4.17.

Table 4.17: Steady State Distributions Different Standard Deviations

	(f_1,f_1,f_1)	(f_1,f_1,f_2)	(f_1,f_2,f_1)	(f_1,f_2,f_2)	(f_2,f_1,f_1)	(f_2,f_1,f_2)	(f_2,f_2,f_1)	(f_2,f_2,f_2)
σ=1.00	0.0709	0.1120	0.1765	0.1406	0.1406	0.1765	0.1120	0.0709
σ=0.50	0.0540	0.1040	0.1984	0.1436	0.1436	0.1984	0.1040	0.0540
σ=0.10	0.0129	0.0647	0.2857	0.1366	0.1366	0.2857	0.0647	0.0129
σ=0.05	0.0033	0.0397	0.3387	0.1183	0.1183	0.3387	0.0397	0.0033
σ=0.01	0	0.002	0.46	0.038	0.038	0.46	0.002	0

4.6.3 Noise and Enforceable Equilibria

In the previous discussion of repeated games and enforceable equilibria, the radios implicitly observed the adaptations of the other radios and punished or rewarded based on those adaptations. However, in the model developed in Chapter 2, the cognitive radios do not directly observe the actions of the other radios and are instead observing the *observable outcomes*, e.g., SINR, throughput, and end-to-end delay, frequently interpreted through the valuations of their utility functions. When each element of the action space is associated with a unique element in the outcome space, i.e., there exists a bijective mapping between the action space and the outcome space, and each player associates each outcome with a unique real number, then it is possible for a player to reason from utilities to actions and correctly assess when a deviation from the agreed strategy has occurred. But the presence of noise violates these assumptions and inhibits the determination of what actions led to their measured utilities.

For instance consider a repeated cognitive radios' dilemma where the radios have agreed to operate at (*n*,*N*) which yields throughputs of (9.6,9.6). Without noise, one radio could detect that the other radio has deviated to a wideband waveform if its throughput dropped to (3.2). If we naively approximate the impact of noise as a zero-mean Gaussian process added to the utility functions, then the maximum likelihood threshold for determining that deviation has occurred is a measured throughput of 6.4 or less. With this threshold and assuming that neither radio has actually deviated and a standard deviation of 1.0, a radio will incorrectly assess that the other radio has switched to the wideband waveform with a probability of 6.8714×10^{-4}.

While this low probability makes an erroneous detection seem improbable, consider a similar game being played by hundreds of radios over hundreds of iterations. The probability of having a false alarm in a single iteration is now 6.6% and the probability of having a false alarm from any of the 100 radios over 100 iterations is 99.9% - a virtual certainty.

Now suppose that each radio is employing a grim trigger strategy wherein detection of a wideband signal (whether correctly or erroneously) leads to the radio transmitting its own wideband waveform for the remainder of the iterations. In short order, the other radios will all be transmitting wideband waveforms because they detected the punishing radio transmitting a wideband signal and must punish the punishing radio. Or if the radios are employing tit-for-tat strategies, it is clear that after a single erroneous punishment, the network will cascade to a state where all radios are employing wideband waveforms.

One way to address this problem is to decrease the probability of false alarm. For example, [Srivastava_06a] considers a network of devices which are contributing resources to a network. Unfortunately, the stage game has the property that not contributing resources to the network is an NE. To enforce a more desirable equilibrium, the devices jointly agree to employ a grim trigger strategy to punish any device that does not make a large contribution to the network. To determine when a deviation has occurred, all devices monitor a signal available to all of the devices, in this case aggregate network goodput. Because of noise in the system, this signal is assumed to represent an imperfect mapping of the actions to the monitored goodput values.

[Srivastava_06a] then considers the following three different deviation detection scenarios or "triggering" events: 1) a single detected deviation where the signal is below a threshold, 2) a running average of three consecutive iterations below the threshold, and 3) three consecutive iterations below the threshold. Intuitively, the probabilities of false alarms under these three scenarios are related by $p_1 > p_2 > p_3$ where p_k is the probability of a false alarm in scenario $k \in \{1,2,3\}$. This relationship is reflected in Figure 4.25 which depicts the percentage of simulated runs where false alarms do not trigger a cascade of punishments.

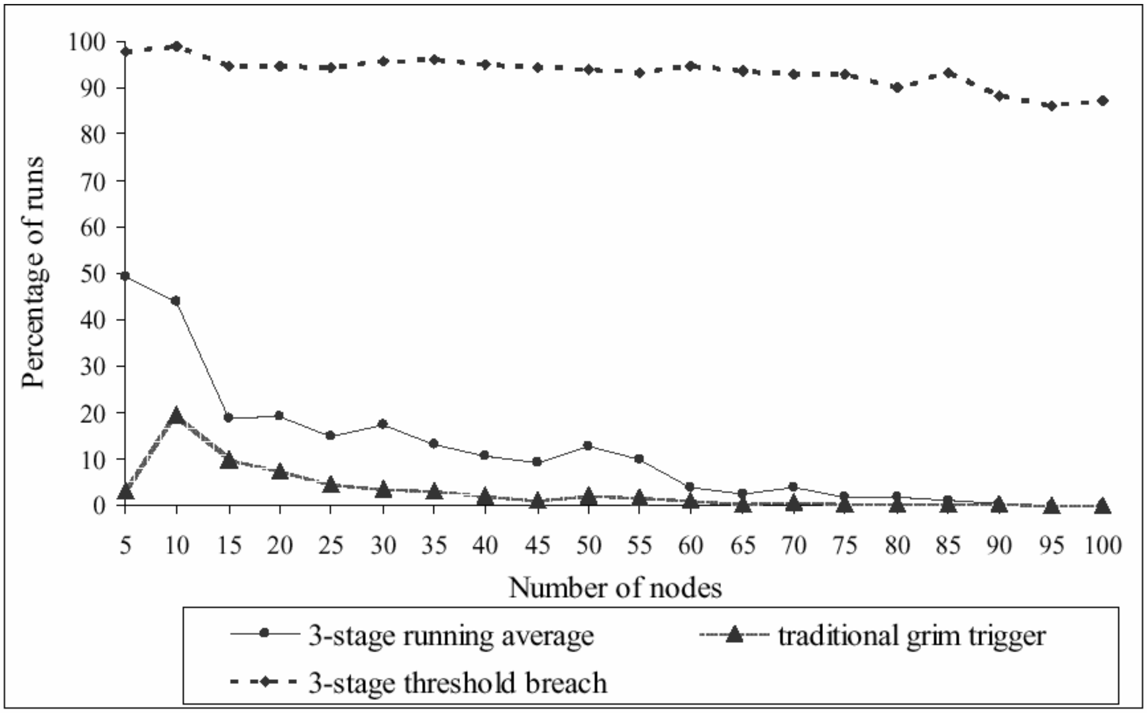

Figure 4.25 Percentage of Runs where a Cascade of Punishments Occurred due to Imperfect Signaling [Srivastava_06a]

Theoretically, all three curves should asymptotically approach zero as with a sufficient number of iterations or radios, the probability of a false alarm asymptotically approaches one for all nonzero probabilities of false alarms. As a cascade of punishments is a catastrophic event for the network, this is a very discouraging result – all such implemented radio networks will eventually meet a catastrophic failure.

One reason that this network terminates in a catastrophic failure is the grim trigger punishment scheme ensures that once the network enters a bad state, it can never recover. Instead of employing the permanent punishments of [Srivastava_06a], the one-off punishment employed by [MacKenzie_01] would seem to limit the immediate impact of a false alarm to a single packet or single event.

However, the other reason that the network of [Srivastava_06a] must fail catastrophically would still be present in one-off punishments of [MacKenzie_01]. Specifically, by only monitoring the public signal, other radios cannot tell differentiate between punishment and selfish deviation. So when one radio in [MacKenzie_01] punishes a false alarm, many other radios will interpret this as a deviation leading to the same catastrophic cascade.

So in addition to Axelrod's suggestion [Axelrod_84] that equilibrium enforcement schemes be "nice", "retaliating", "forgiving", and "non-envious", the network must include some additional mechanism to signal when a radio is engaging in punishment so other radios can differentiate between selfish deviations and punishment. Note that all of these properties should be present. For example, even with a "punishment signal" a unforgiving scheme perhaps with the grim triggers in [Srivastava_06a] will still eventually converge (albeit more slowly) to the catastrophic network. However, the coupling of a punishment signal with the forgiving scheme of [MacKenzie_01] should never lead to a catastrophic network if the network starts at the agreed state (nice), properly incentivizes the cooperative strategy (retaliating) and the radios do not attempt to ensure that they outperform other radios (non-envious).

In addition to the implicit additional complexity of ensuring that punishment is not confused with deviations [Neel_04b], this signal will also be fraught with errors and, worse, may be subject to radios gaming this "punishment signal". Fortunately, the gaming of the punishment signal could largely be eliminated for weak CRs which are constrained by their programming. Thus for weak CR, gaming of the punishment signal could be eliminated by reviewing the radio's programming perhaps as part of a device testing and certification process. However, a strong CR, which is not constrained by its programming, cannot be guaranteed to not misuse the punishment signal.

4.7 Analysis Summary and Design Implications

This chapter introduced several powerful techniques for analyzing the interactions of ontological and procedurally random cognitive radios based on knowledge of the radios' goals, actions, and interactions and broad assumptions about the radios' decision rules, e.g., that the radios act in their own self interest in response to their observations. While most of these techniques presented were originally developed in game theory to study the interactions of humans, they are surprisingly useful for analyzing the interactions of cognitive radio networks, though with some difficulties when ascertaining optimality and the impact of noise. It was also seen that it is possible to incorporate approaches from both game theory and the traditional engineering techniques of Chapter 3 to yield powerful insights, such as the convergence results of Section 4.5 which used Markov chain concepts from Chapter 3 and stage game properties to assess the convergence criteria of broad classes of decision algorithms under various decision timings.

These analysis insights are summarized in Section 4.7.1 and then applied as design guidelines for cognitive radio in Section 4.7.2.

4.7.1 Analysis Summary

The cognitive radio network modeled in Chapter 2 by the tuple $\langle N, A, \{u_i\}, \{d_i\}, T\rangle$ can be seamlessly recast as a game with N as the players (radios), A as the action space (adaptations), $\{u_i\}$ as the utility functions (goals), $\{d_i\}$ as the decision rules, and T as the decision timings. The normal form game which models a single-shot synchronous adaptation by all players is given by the tuple $<N, A, \{u_i\}>$. The repeated game $\langle N, A, \{u_i\}, \{d_i\}\rangle$ models cognitive radio networks that adapt repeatedly with synchronous timing and is especially well suited for networks where radios incorporate punishment and reward strategies. The myopic repeated game $\langle N, A, \{u_i\}, \{d_i\}, T\rangle$ models scenario where radios adapt to the most recent state of the network under a variety of different decision timings. The mixed strategy game, $\Gamma' = \langle N, \Delta(A), \{U_i\}\rangle$, models scenarios where radios can probabilistically play different waveforms.

4.7.1.1 Steady States

The primary steady-state concept introduced in this chapter was the Nash Equilibrium (NE) which is a fixed point for all self-interested myopic decision rules. An NE is known to always exist for finite normal form games that are IESDS solvable (Algorithm 4.2), have FIP (Theorem 4.7), or have weak FIP. In infinite games where action sets are convex and compact, an NE exists if the utility functions are continuous in a and quasi-concave in a_i (Theorem 4.3). For mixed extensions to finite normal form games, a mixed strategy NE always exists (Theorem 4.4). While identifying these equilibria can require exhaustive searches for finite games, the unique NE for an IESDS can be found by

Algorithm 4.2 and the NE for games with continuous best responses can be found by simultaneously solving (4.7) $\forall\ i \in N$.

It was seen that repeated games with punishment where players incorporate discounted future payoffs can enforce an infinite number of different equilibria (Theorem 4.5).

4.7.1.2 Optimality

This chapter presented the concept of Pareto Optimality (Definition 4.9), a traditional technique use in game theory to assess optimality. It was seen that when a cognitive radio network designer already has a particular objective function in mind, the designer would be better off ignoring the Pareto optimality of the network's steady-states due to the computational cost, imprecision, and potentially misleading results yielded by evaluating Pareto optimality. However, when the network designer does not have a particular objective for the network in mind, then Pareto optimality is a reasonable criteria to assess network performance.

4.7.1.3 Convergence

This chapter particularly focused on the convergence of myopic repeated games wherein radios adapt their waveforms based on their most recent observations. In order for play in these games to guaranteed to converge, it was seen the stage game needed to exhibit special properties such as IESDS solvability, weak FIP, or FIP. It was seen that for finite games, the widest set of convergence conditions held when at least one of the following circumstances held: the cognitive radios incorporate randomness into their decision processes, the network exhibits random or asynchronous decision timings, or the stage game has FIP.

4.7.1.4 Noise

It was seen that noise can significantly impact the behavior of cognitive radio networks. While game theory typically treated errors in play as being implementation errors – a reasonable assumption for humans – the source of errors for cognitive radios is more likely to be caused by observations being corrupted by noise. Regardless of the source, these errors lead to the Markov models of the networks changing from absorbing Markov chains to ergodic Markov chains. This ultimately has the meaning that for networks that can be modeled as myopic repeated games, the presence of noise means that the network has a theoretically nonzero chance of passing through every possible network state. However, the original absorbing states tend to remain the most commonly visited states in the network. So even with noise in the system, the Nash equilibrium concept (absorbing states for games with weak FIP) retain significant power for predicting the state of the network.

For networks incorporating punishment strategies, it was seen that noise can have a catastrophic impact on the performance of a network. Specifically, a cascade of punishments can be spawned by a single misdetection of a deviation, the probability of which asymptotically approaches 1 for networks that run over thousands of iterations. Rather than simply improving deviation detection algorithms, it was seen that it was necessary to incorporate forgiveness and some means to signal that punishment is occurring.

4.7.2 Design Implications

While numerous game models were considered, this chapter effectively considered two broad approaches for implementing cognitive radios.

- *Loner radios* – radios that myopically choose actions to improve their own performance as part of a distributed network of autonomous radios.
- *Social radios* – radios that, along with other radios in the network, enforce an equilibrium condition on all radios in the network by adapting their actions so as degrade the performance of radios that deviate from the equilibrium or improve the performance of radios that adhere to the equilibrium.

4.7.2.1 Loner Radios

A loner radio is a myopic autonomous cognitive radio that chooses its actions without specific regard to how its adaptations will influence the adaptations of other radios. Their adaptations are guided solely by their own most recent observations of their own performance. More generally, these observations may be informed by client nodes or a receiver, but these cognitive radios are not explicitly considering other cognitive radios' actions nor their utilities. In general, loner radios are the simplest to implement as they do not have to be aware of how their behavior is influencing other radios and can frequently be implemented with very simple decision algorithms.

A network of loner radios could be modeled as a myopic repeated game. As such the network cannot be assured to have a steady-state, nor to be optimal, nor to converge. However, the existence of a steady state and convergence conditions can be assured if it can be shown that the radios' goals and actions form a normal form game has special that is IESDS solvable or has FIP or weak FIP. When these conditions are satisfied, broad classes of self-interested myopic algorithms are known to converge to an NE and that the absorbing states remain the most likely states even when noise corrupts the processes.

If the NE are also optimal, then loner radio networks with FIP or weak FIP can realize what would otherwise be assumed as a panglossian scenario – low complexity radios that place no additional demands on the network, such as decision coordination and distribution, yet autonomously reason their way to a desirable state with relatively simple decision rules. The key to realizing this best possible of all possible cognitive radio networks will be establishing when the loner radios' goals and available adaptations can be modeled as a game with weak FIP or FIP. Because of the great promise of this approach, the remaining chapters of this work focus on establishing when a cognitive radio network can be modeled as having FIP or weak FIP and developing applications that leverage this knowledge.

4.7.2.2 Social Radios

Unlike a loner radio, a social radio takes into specific consideration the actions and utilities of other radios in the network. With this knowledge, a social radio can fashion algorithms that utilize combinations of punishment and reward to enable the network to enforce the large number of equilibria permitted by the Folk Theorem.

With social radios, it is relatively trivial to ensure that a network of social radios operates at an optimal equilibrium – simply specify that action tuple as the operational state and have the radios enforce the state – and without noise, convergence to that state can be trivially assumed. Further, the social radio approach can theoretically be leveraged to achieve optimal performance for any operational scenario, and convergence is implicit to the punishment and reward with no further analysis required to theoretically know in what state the network will operate.

However, there are several practical considerations that must be addressed when implementing social radios. First, social radios must agree on the state that should be enforced. This either implies that the cognitive radios are capable of negotiating this state among themselves or that the network includes some authority that sets the operational state. Either solution would expect heavy usage as the optimal solution will be constantly changing as radios enter and leave the network, multipath profiles change, mobility alters network topologies, and user applications change.

Second social radios must have some means of gathering the requisite actions and utilities of the other radios. As directly observing the actions of other social radios may be very difficult, a radio may need to be able to infer other radios' actions from its own utility function. Alternately, the network may need to include some mechanism for distributing this information, perhaps via a dedicated channel or via calls to a Radio Environment Map [Zhao_06].

Third, it will be important for social radios to be able to differentiate between punishment and deviation as the failure to differentiate will cause any punishment scheme to converge to a catastrophic network. Again this problem could be solved through a dedicated channel or through calls to a Radio Environment Map [Zhao_06].

Finally, it will also be necessary to carefully understand the tradeoffs in the design of the punishment, reward, and detection schemes, and how those schemes impact the performance of the network. In general, the recommendations of [Axelrod_84] and this document note that these schemes, should be nice, retaliating, forgiving, and non-envious, with discriminating diction. But when determining specifics, the optimal point in the detection/false-alarm space is a function of the impact of erroneous punishment/reward with the operational environment influencing both. Thus, while a social radio approach should be generally applicable to any operational scenario, the optimal design of social radio algorithms will vary by operational scenario implying that significant analysis work should also be performed before fielding a social radio network.[18]

So while a network of social radios may simplify the requisite analysis required to ensure convergence and optimality as compared to a network of loner radios, analysis must still be performed. Further proper operation of a social radio network necessitates additional signaling mechanisms to support negotiation, distribution of action and utility information, and to differentiate between deviations and punishments.[19] It also generally, though not always, implies greater complexity in the radio itself. While addressing these various issues is beyond the scope of this document, they likely will be addressed as part of the follow-on work to [Srivastava_06b].

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[18] Alternately to analysis of either social or loner radio networks, cognitive radio networks it may be possible to just allow cognitive radio networks to "evolve" under the proper guidelines. It is anticipated that the techniques for properly guiding this evolution could be inferred from evolutionary game theory – a topic beyond the scope of this document.

[19] In specific instances, it may be possible for social radios to agree to punish using only one action. Any action other than the punishment action or an action in the enforced equilibrium could then be interpreted as a deviation. However, with noise this approach would still be susceptible to errors.

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Chapter 5: Potential Games

“All roads lead to Rome.” - proverb

Suppose this proverb were literally true. Then starting from any place in the world, you could start walking down any road, and assuming you picked the right direction (if all roads lead to Rome, then all roads also lead away from Rome), you would eventually end up in Rome. Further, suppose that during your travel you came to an intersection, you could choose to follow any road at the intersection and again assuming you picked the right direction, you would still end up in Rome. Indeed this conclusion would hold for any number of intersections. Now suppose that during your travel you occasionally make a mistake choosing a direction at an intersection. Assuming that you make a limited number of mistakes, you would still arrive in Rome. Further suppose that once in Rome you accidentally wander out of Rome. Again assuming you generally pick the right direction on each road, you’ll come back to Rome.

Of course, not all roads lead to Rome – just consider driving to Rome from Virginia! But this maxim is reminiscent of a normal form game property defined in Chapter 4 - the *Finite Improvement Property* (*FIP*). A game has FIP if all improvement paths are finite, or less precisely, if players follow an improvement path for enough iterations, they will terminate at a Nash Equilibrium (NE). The maxim does differ somewhat from FIP as a game with FIP may have multiple equilibria. Then again, according to Dr. Allen MacKenzie, there is a saying at the University of Virginia that all *dirt* roads lead to Blacksburg. So perhaps there’s room for multiple equilibria in the maxim as well.

While Chapter 4 showed that FIP was a very strong property – implying convergence of many self-interested cognitive radio algorithms, determining when a game has FIP was a daunting task – either an exhaustive listing of improvement paths or an exhaustive search for improvement cycles. Overcoming this difficulty, this chapter presents a class of readily identified games known as *potential games* that have FIP. In fact, one particular class of potential game – *generalized ordinal potential games* – is coincident with the set of games that have FIP.

Beyond the implications of FIP covered in Chapter 4, potential games also satisfy the conditions of some of the more powerful concepts from Chapter 3, specifically Zangwill's convergence theorem, and Lyapunov stability, bringing procedural radio analysis techniques to ontological radios. Using these theorems will enable us to extend the convergence and noise results of Chapter 4 to games with infinite action sets.

Because of the ease of verifying that a cognitive radio algorithm satisfies the conditions of a potential game (as simple as evaluating a second-order derivative), the readily identified equilibria, and the broad class of low complexity algorithms are guaranteed to converge to stable equilibria, the potential game model is particularly attractive for the design of cognitive radio algorithms. The primary limitation of designing cognitive radio algorithms with potential games is that loner radio networks cannot be assured of desirable performance. However, Chapter 6 introduces some additional conditions to the potential game concept that ensure that loner radio networks realize desirable behavior from selfish adaptations.

This chapter focuses on developing techniques for determining if a cognitive radio network and establishing the performance implications Subsequent chapters leverage these results to develop powerful cognitive radio algorithms. In the remainder of this chapter, Section 5.1 formalizes the concept of the potential game model and establishes the connection between FIP and potential games enabling us to apply the results of Chapter 4. Section 5.2 discusses how potential games can be identified and specifically how a cognitive radio needs to be designed to be a potential game. Section 5.3 presents a number of special properties of potential games that aid the design and analysis of cognitive radios. Then addressing our four analysis objectives, Section 5.4 discusses the steady-state properties of potential games, Section 5.5 discusses the desirability properties of potential games, Section 5.6 presents conditions under which decision rules converge, and Section 5.7 considers the impact of noise on potential game networks. Section 5.8 concludes with a discussion of the use of potential games in the wireless and cognitive radio literature and the impact of potential games on the modeling, analysis, and design of cognitive radio networks.

5.1 Potential Games

As formalized in [Monderer_96], a potential game is a normal form game which has the property that there exists a function known as the *potential function*, $V:A\to\mathbb{R}$, that reflects the change in value accrued by every unilaterally deviating player. This concept can be refined into the following five fundamental classes of potential games: *exact potential games*, *weighted potential games*, *ordinal potential games*, *generalized ordinal potential games*, and *generalized e-potential games*. Under typical conditions, these games satisfy Zangwill's convergence theorem and exhibit FIP and a related property known as Asymptotic Finite Improvement Property. In this section, these classes games are formally defined and categorized and the relationship between potential games and FIP is formalized.

5.1.1 Potential Game Definitions

This section defines exact, weighted, ordinal, generalized ordinal, and generalized ε-potential games and provides examples of each of these games.

5.1.1.1 Exact Potential Games

Definition 5.1: *Exact Potential Game*

A normal form game, $\Gamma=\langle N, A, \{u_i\}\rangle$, is said to be an *exact potential game* if there exists a function, $V:A\to\mathbb{R}$, known as an *exact potential function*, that satisfies $u_i(b_i,a_{-i})-u_i(a_i,a_{-i})=V(b_i,a_{-i})-V(a_i,a_{-i})\ \forall i\in N, \forall a\in A$.

For everywhere differentiable utility functions, an equivalent condition is the existence of an exact potential function, V, which satisfies (5.1) $\forall i\in N, \forall a\in A$.

$$\frac{\partial u_i(a)}{\partial a_i}=\frac{\partial V(a)}{\partial a_i} \tag{5.1}$$

Example 5.1: A 2x2 Exact Potential Game

Consider the prisoners' dilemma variant shown below in matrix representation labeled as Γ and the function V that accompanies Γ.

Γ	*A*	*B*
a	(3,3)	(0,5)
b	(5,0)	(1,1)

$V(\cdot)$	*A*	*B*
a	0	2
b	2	3

Examining Table 5.1, which provides a listing of all profitable unilateral deviations in Γ (all unprofitable unilateral deviations can be found by reversing the direction of the deviations listed in the left column), we see that $u_i(b_i, a_{-i}) - u_i(a_i, a_{-i}) = V(b_i, a_{-i}) - V(a_i, a_{-i}) \quad \forall i \in N, \forall a \in A$, thereby satisfying the conditions of Definition 5.1.

Table 5.1: Unilateral Deviation Relationships for Γ and V

	Unilateral Deviation	Change in utility for deviating player	Change in V
Row Player	(a,A)⇒(b,A)	3⇒5 (+2)	0⇒2 (+2)
	(a,B)⇒(b,B)	0⇒1 (+1)	2⇒3 (+1)
Column Player	(a,A)⇒(a,B)	3⇒5 (+2)	0⇒2 (+2)
	(b,A)⇒(b,B)	0⇒1 (+1)	2⇒3 (+1)

There are a couple of additional insights that may be gleaned from this example. First, the definition of an exact potential game does not imply differential equality condition for multilateral deviations as (a,A)⇒(b,B) reduces the utility for both players (3⇒1) while *V* actually increases (0⇒3). Simply because an action tuple maximizes *V* (making the action tuple an NE), it does not mean that the action tuple is optimal or desirable as evidenced by the fact that the NE is the only point that is not Pareto efficient. Second, this exact potential function is not unique. In fact any number of other exact potential functions for Γ can be found as $V' = V + c$, where c is an arbitrary real constant added to all elements in the range of *V*. For instance, the function given below, *V'*, is also an exact potential function for Γ and is given by $V' = V - 1$.

V'(·)	A	B
A	-1	1
B	1	2

5.1.1.2 Weighted Potential Games

A weighted potential relaxes the conditions on the exact potential game so that the differential equality relationship is scaled by a weight for each player. More formally, a weighted potential game can be defined as shown in Definition 5.2.

Definition 5.2: *Weighted Potential Game*
A normal form game, $\Gamma = \langle N, A, \{u_i\} \rangle$, is said to be a *weighted potential game* if there exists some function, $V : A \to \mathbb{R}$, known as a *weighted potential function*, that satisfies $u_i(b_i, a_{-i}) - u_i(a_i, a_{-i}) = \alpha_i [V(b_i, a_{-i}) - V(a_i, a_{-i})]$ $\forall i \in N, \forall a \in A$, $\alpha_i > 0$

Like the exact potential game, an equivalent formulation for a weighted potential game exists when the utility functions are everywhere differentiable. Specifically, a normal form game is a weighted potential function if there exists a function V such that (5.2) is satisfied $\forall i \in N, \forall a \in A$.

$$\frac{\partial u_i(a)}{\partial a_i} = \alpha_i \frac{\partial V(a)}{\partial a_i} \tag{5.2}$$

Example 5.2: A 2x2 Weighted Potential Game

Slightly modifying the payoffs of the game in Example 5.1, consider the two player normal form game, Γ, shown in matrix representation below which is accompanied by the function *V*.

Γ	A	B
a	(3,3)	(0,5)
b	(7,0)	(2,1)

V(·)	A	B
a	0	2
b	2	3

Table 5.2 tabulates all profitable unilateral deviations in Γ (all unprofitable unilateral deviations can be found by reversing the direction of the deviations listed in the left column). Note that for unilateral deviations by the row player, the change in V and the change in utility are no longer exactly equal. However, if V is scaled by a factor of 2,

equality is again achieved. Thus, this game is a weighted potential game with weights $a_{row} = 2$ and $a_{col} = 1$ thereby satisfying the conditions of Definition 5.2.

Table 5.2: Unilateral Deviation relationships for Γ and *V*

	Unilateral Deviation	Change in utility for deviating player	Change in V	α_j
Row Player	(a,A)⇒(b,A)	3⇒7 (+4)	0⇒2 (+2)	α_{row}=2
	(a,B)⇒(b,B)	0⇒2 (+2)	2⇒3 (+1)	
Column Player	(a,A)⇒(a,B)	3⇒5 (+2)	0⇒2 (+2)	α_{col}=1
	(b,A)⇒(b,B)	0⇒1 (+1)	2⇒3 (+1)	

Note that this modified game preserves the preference relations, improvement paths, and potential function of the previous example's game. Because of this, these two games are said to be *better-response equivalent* – a concept considered more fully in Section 5.2.

5.1.1.3 Ordinal Potential Games

If we further relax the conditions on the relationship between *V* and the utility functions so that only sign changes are preserved, then we have an ordinal potential game, more formally defined in Definition 5.3.

Definition 5.3: *Ordinal Potential Game*
A normal form game, $\Gamma = \langle N, A, \{u_i\} \rangle$ is said to be an *ordinal potential game* if there exists some function, $V : A \rightarrow \mathbb{R}$, known as the *ordinal potential function*, that satisfies $u_i(b_i, a_{-i}) - u_i(a_i, a_{-i}) > 0 \Leftrightarrow V(b_i, a_{-i}) - V(a_i, a_{-i}) > 0 \; \forall i \in N, \forall a \in A$.

Like the previous potential games, an equivalent formulation exists for everywhere differentiable utility functions and is satisfied when there exists a function, $V : A \rightarrow \mathbb{R}$ that satisfies (5.3) $\forall i \in N, \forall a \in A$.[1]

$$\operatorname{sgn}\left\{\frac{\partial u_i(a)}{\partial a_i}\right\} = \operatorname{sgn}\left\{\frac{\partial V(a)}{\partial a_i}\right\} \tag{5.3}$$

Interestingly, if a game is an exact or weighted potential game with differentiable utility functions, then its potential function must also be differentiable. However, if a game is an

[1] It is believed that this differentiable definition of an ordinal potential game is a novel result, but this is unclear as it is a rather obvious extension of the differentiable definitions of exact and weighted potential games.

ordinal potential game with differentiable utility functions, then the potential function need not be differentiable.

Example 5.3: A 2x2 Ordinal Potential Game

Consider the normal form game, Γ, shown in matrix representation below with accompanying function *V*.

Γ	*A*	*B*
a	(1,-1)	(2,0)
b	(2,0)	(0,1)

V(·)	*A*	*B*
a	0	3
b	1	2

Table 5.3 tabulates all profitable unilateral deviations in Γ (all unprofitable unilateral deviations can be found by reversing the direction of the deviations listed in the left column). Based on the listed relationships, Γ is neither a weighted nor an exact potential game. However, the sign is always preserved (increases in utility are reflected in increases in *V*) making this game an ordinal potential game.

Table 5.3: Unilateral Deviation relationships for Γ and *V*

	Unilateral Deviation	Change in utility for deviating player	Change in V
Row Player	(*a*,*A*)⇒(*b*,*A*)	1⇒2 (+1)	0⇒1 (+1)
	(*b*,*B*)⇒(*a*,*B*)	0⇒2 (+2)	2⇒3 (+1)
Column Player	(*a*,*A*)⇒(*a*,*B*)	-1⇒0 (+1)	0⇒3 (+3)
	(*b*,*A*)⇒(*b*,*B*)	0⇒1 (+1)	1⇒2 (+1)

Note that this game, like the previous two games has the FIP property and that the potential function is monotonically increasing along each improvement path. In general, there is a close relationship between the FIP property and potential games, a relationship clarified in the following.

5.1.1.4 Generalized Ordinal Potential Games

While the ordinal potential game definition seems closely related to FIP, not every game with FIP is an ordinal potential game. Consider the 2x2 game represented in matrix form shown in Figure 5.1 which was originally presented in [Monderer_96] and is quite similar to the previous example.

Γ	*A*	*B*
A	(1,0)	(2,0)
B	(2,0)	(0,1)

Figure 5.1: A Game with FIP but no Ordinal Potential [Monderer_96]

This game can be readily verified to have FIP by listing all improvement paths as done in Table 5.4. However, for this game to have an ordinal potential, the following relationships would have to be satisfied – $V(a, A) < V(b, A) < V(b, B) < V(a, B) < V(a, A)$ – a logical impossibility.

Table 5.4: Improvement Paths for Game in Figure 5.1

$g_1 = ((a, A), (b, A))$	$g_3 = ((b, B), (a, B))$	$g_5 = (g_2, g_3)$
$g_2 = ((b, A), (b, B))$	$g_4 = (g_1, g_2)$	$g_6 = (g_1, g_5)$

While not all games with FIP are ordinal potential games, all games with FIP do belong to a broader class of games known as *generalized ordinal potential games*.[2]

<u>Definition 5.4</u>: *Generalized Ordinal Potential Game*
A normal form game, $\Gamma = \langle N, A, \{u_i\} \rangle$ is said to be a *generalized ordinal potential game* if there exists some function, $V : A \rightarrow \mathbb{R}$, known as the *generalized ordinal potential function*, that satisfies $u_i(b_i, a_{-i}) > u_i(a_i, a_{-i}) \Rightarrow V(b_i, a_{-i}) > V(a_i, a_{-i}) \quad \forall i \in N, \forall a \in A$.

Unsurprisingly, it can be maddeningly difficult to identify games that are generalized ordinal potential games but are not also ordinal, weighted, or exact potential games. Reviewing the game in Figure 5.1 we can verify that Figure 5.2 is a generalized ordinal potential function for the game. Note that while $u_i(b_i, a_{-i}) > u_i(a_i, a_{-i}) \Rightarrow V(b_i, a_{-i}) > V(a_i, a_{-i})$ is true, the following relationship, which would be required for Γ to be an ordinal potential game, fails: $u_{col}(a, A) > u_{col}(a, B) \Leftarrow V(a, A) > V(a, B)$.

$V(\cdot)$	*A*	*B*
a	0	3
b	1	2

Figure 5.2: Generalized Ordinal Potential Function for Game in Figure 5.1

[2] For the moment, we assert this property without proof. However, a proof of this property is provided in Section 5.3.1.

5.1.1.5 Generalized e-Ordinal Potential Games

We present one more class of potential games, *generalized e-potential games*, which are especially useful for establishing convergence of potential games with infinite action sets.

> **<u>Definition 5.5</u>**: *Generalized e-Potential Game* (*)
> A normal form game, $\Gamma = \langle N, A, \{u_i\} \rangle$ is said to be a *generalized e- potential game* if there exists some function, $V : A \rightarrow \mathbb{R}$, known as the *generalized e-potential function*, that given $e_1 > 0$ there is an $e_2 > 0$ such that $u_i(b_i, a_{-i}) > u_i(a_i, a_{-i}) + e_1 \Rightarrow V(b_i, a_{-i}) > V(a_i, a_{-i}) + e_2 \; \forall i \in N, \forall a \in A$.

It may be that the sets of generalized ordinal potential games and generalized ε-potential games are coincident. However, the author does not know of a proof to that effect, nor has he identified a promising approach for proving such a connection. Lacking a clearly defined connection, this document treats these two concepts independently.

5.1.2 Relationships between Potential Game Classes

By applying the definitions of the preceding section it is readily apparent that every exact potential game is a weighted potential game (where $a_i = 1 \forall i \in N$); every weighted potential game is an ordinal potential game; and every ordinal potential game is a generalized ordinal potential game. Further every weighted potential game is also a generalized ε-potential game.

> **<u>Theorem 5.1</u>**: *Weighted and Generalized e-Potential Games* (*)
> If $\Gamma = \langle N, A, \{u_i\} \rangle$ is a weighted potential game, then Γ is also a generalized ε-potential game.
> *Proof*: As $u_i(b_i, a_{-i}) - u_i(a_i, a_{-i}) = a_i [V(b_i, a_{-i}) - V(a_i, a_{-i})] \; \forall i \in N$,
> $u_i(b_i, a_{-i}) > u_i(a_i, a_{-i}) + e_1 \Rightarrow V(b_i, a_{-i}) > V(a_i, a_{-i}) + a_i e_1$. Setting $e_2 = \min_{i \in N}\{a_i e_1\}$ then supplies the requisite $e_2 > 0$ for Definition 5.5.

Considering the set of exact potential games, *EPG*, the set of weighted potential games, *WPG*, the set of ordinal potential games, *OPG*, and the set of generalized ordinal potential games, *GOPG*, the relationships between these sets can be visualized using the Venn diagram shown in Figure 5.3.

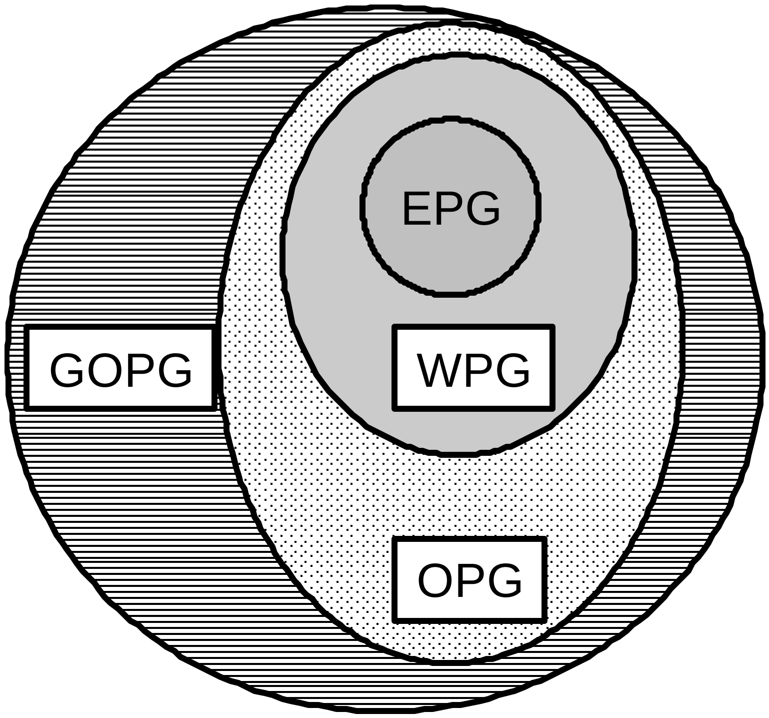

Figure 5.3: Potential Game Venn Diagram. Adapted from Fig.1 in [Voorneveld_00]

For the remainder of this chapter, we use the term "potential game" to refer to a generalized ordinal potential game. In general this nomenclature will be quite useful as an exact potential game is a weighted potential game is an ordinal potential game is a generalized ordinal potential game and all generalized ordinal potential games have FIP. However, when a particular property is restricted to a subclass of potential games, then the term that corresponds to the broadest class of applicable potential games is used.

5.2 Identification Techniques

While the definitions introduced in Section 5.1.1 can be used to show that a particular game is an exact potential game, this approach requires a seeming *deus ex machina* introduction of a potential function. This section presents various techniques by which exact and ordinal potential functions can be shown to exist and by which exact and ordinal potential functions can be identified. This section also includes a review of classes of games that can be shown to be exact potential games and examples of games frequently encountered in the field of economics applicable to cognitive radio networks that can be shown to be exact potential games. Throughout the discussion in this section it is assumed that we are working with a normal form game defined as $\Gamma = \langle N, A, \{u_i\} \rangle$.

5.2.1 Exact Potential Game Identification

For normal form games with utility functions that are everywhere twice differentiable, there exists a simple technique for determining whether or not the game is an exact potential games.

5.2.1.1 Twice Continuously Differentiable EPG Existence

When all $u_k \in \{u_i\}$ are twice continuously differentiable and V is a EPF, [Monderer_96] states that (5.4) must hold $\forall i, j \in N, \forall a \in A$. Further, (5.5) constitutes a sufficient condition for the existence of a potential function.

$$\frac{\partial^2 u_i(a)}{\partial a_i \partial a_j} = \frac{\partial^2 u_j(a)}{\partial a_j \partial a_i} = \frac{\partial^2 V(a)}{\partial a_i \partial a_j} \tag{5.4}$$

$$\frac{\partial^2 u_i(a)}{\partial a_i \partial a_j} = \frac{\partial^2 u_j(a)}{\partial a_j \partial a_i} \tag{5.5}$$

When (5.5) is satisfied, [Monderer_96] gives the following equation for finding the potential function.

$$V(a) = \sum_{i \in N} \int_0^1 \frac{\partial u_i}{\partial a}(x(t)) x_i'(t)\, dt \tag{5.6}$$

where x is a piecewise continuously differentiable path that connects some fixed action tuple b to some other action tuple a such that $x:[0,1] \rightarrow A$ ($x(0) = b$, $x(1) = a$). Although quite general, the evaluation of (5.6) can be tedious. Frequently, the solution of a potential function is more easily accomplished by demonstrating that the game is one of a handful of common exact potential game forms and then applying an associated equation to find the exact potential function.

5.2.1.2 Common Exact Potential Game Forms

In the following we show that the following are exact potential games: coordination-dummy games, coordination games, dummy games, bilateral symmetric interaction games, and multilateral symmetric interaction games. While some of the considered games are trivial, more complex exact potential can be formed through the linear combination of exact potential games as shown in Section 0.

5.2.1.2.1 Coordination-Dummy Games

If all players in a game have an objective function that can be characterized as in (5.7)

$$u_i(a) = C(a) + D_i(a_{-i}) \tag{5.7}$$

where $C: A \to \mathbb{R}$ and $D_i: A_{-i} \to \mathbb{R}$, then the game is said to be a *coordination-dummy* game. The potential for this game can be written as $V(a) = C(a)$.

$C(a)$ defines what is known as a coordination (identical-interest) function wherein all players receive the same payoff for a particular action tuple a, and $D_i(a_{-i})$ defines a dummy function – a function where the outcome for player i is not dependent on the actions of i and is instead solely a function of the actions of the other players. Note that each player in the coordination-dummy game may have its own dummy function.

It is relatively easy to show that $C(a)$ is an exact potential for this game by noting that $u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) = C(a_i, a_{-i}) - C(b_i, a_{-i}) \forall i, j \in N, \forall a \in A$. In fact all exact potential games can be expressed as coordination-dummy games.

<u>Theorem 5.2</u>: *Exact Potential Games, Coordination Games, and Dummy Games*
Γ is an exact potential game if and only if there exist functions $C: A \to \mathbb{R}$ and $D_i: A_{-i} \to \mathbb{R}$ such that $u_i(a) = C(a) + D_i(a_{-i})\ \forall i \in N, a \in A$.
Proof:
$\Rightarrow$ Sufficiency was established in the preceding discussion.
$\Leftarrow$ Suppose there is no $C: A \to \mathbb{R}$ such that $u_i(b_i, a_{-i}) - u_i(a_i, a_{-i}) = C(b_i, a_{-i}) - C(a_i, a_{-i})\ \forall i \in N$. Then no exact potential exists for the game per Definition 5.1. Therefore there must be a $C: A \to \mathbb{R}$ such that $u_i(b_i, a_{-i}) - u_i(a_i, a_{-i}) = C(b_i, a_{-i}) - C(a_i, a_{-i})\ \forall i \in N$ and a potential is given by $V = C$.

Example 5.4: Coordination-Dummy Game

Consider the two player normal form game, Γ, shown in matrix representation in Figure 5.4.

Γ	*A*	*B*
A	(1,0)	(3,3)
B	(4,4)	(0,1)

Figure 5.4 Relaxed Coordination Game

This game can be equivalently expressed as a coordination-dummy game with the following coordination, $C(\cdot)$, and dummy, $D(\cdot)$, functions.

$C(\cdot)$	A	B
a	0	3
b	3	0

$D(\cdot)$	A	B
a	(1,0)	(0,0)
b	(1,1)	(0,1)

5.2.1.2.2 Weighted Coordination-Dummy Game

A similar game can be introduced for a weighted potential game wherein all players' utility functions are of the form (5.8).

$$u_i(a) = \alpha_i C(a) + D_i(a_{-i}) \tag{5.8}$$

where $C: A \to \mathbb{R}$, $D_i: A_{-i} \to \mathbb{R}$, and $\alpha_i \in \mathbb{R}$. This can be readily verified as a weighted potential game by applying Definition 5.2. Using (5.8), a theorem similar to Theorem 5.2 can be established.

<u>Theorem 5.3</u>: *Weighted Potential Game Existence*
Γ is a weighted potential game if and only if there exist functions $C: A \to \mathbb{R}$ and $D_i: A_{-i} \to \mathbb{R}$ and scalars $\alpha_i \in \mathbb{R}$ such that $u_i(a) = \alpha_i C(a) + D_i(a_{-i})\ \forall i \in N, a \in A$.
Proof:
$\Rightarrow$ Sufficiency was established in the preceding discussion.
$\Leftarrow$ Suppose it is not possible to identify a function C such that
$u_i(b_i, a_{-i}) - u_i(a_i, a_{-i}) = \alpha_i \left[C(b_i, a_{-i}) - C(a_i, a_{-i})\right]\ \forall i \in N$. Then no weighted potential exists for the game per Definition 5.2.

Note, an equivalent formulation to Theorem 5.3 is given by Theorem 2.1 in [Fachini_97].

5.2.1.2.3 Coordination Games

If all players have utility functions given by (5.9)

$$u_i(a) = C(a) \tag{5.9}$$

where $C: A \to \mathbb{R}$, then the game is said to be a *coordination game*. This is just a coordination-dummy game where $d_i = 0\ \forall i \in N$ and thus is an exact potential game with potential function C. A similar game – a weighted coordination game - can also be constructed wherein $u_i(a) = \alpha_i C(a)$ where $\alpha_i \in \mathbb{R}$ and can similarly be shown to be a weighted potential game.

5.2.1.2.4 Dummy Games

A dummy game is just a coordination-dummy game where $C(a)=0 \forall a \in A$. Note that the potential function for any dummy game is any arbitrary constant function, i.e., $V(a)=c\ \forall a \in A$ where $c \in \mathbb{R}$.

5.2.1.2.5 Self-Motivated Games

If all players have utility functions given by

$$u_i(a)=S_i(a_i) \tag{5.10}$$

where $S_i : A_i \to \mathbb{R}$, then this "game" is said to be a *self-motivated game*. The term game is placed in quotation marks as we have defined a game to be a model of an interactive decision process. For a self-motivated game, there is no interaction and thus the term game is technically inaccurate. A potential function for a self-motivated is given by the expression: $V(a)=\sum_{i \in N} S_i(a_i)$. However, the concept of a self-motivated game is useful for analysis of additive potential games, a topic explored in Section 5.3.

5.2.1.2.6 Bilateral Symmetric Interaction (BSI) Games

As introduced in [Ui_00], if every player's utility function can be characterized by (5.11)

$$u_i(a)=\sum_{j \in N \setminus \{i\}} w_{ij}(a_i,a_j)-S_i(a_i) \tag{5.11}$$

where $w_{ij} : A_i \times A_j \to \mathbb{R}$ and $S_i : A_i \to \mathbb{R}$ such that for every $(a_i,a_j) \in A_i \times A_j$, $w_{ij}(a_i,a_j)=w_{ji}(a_j,a_i)$, then the game is a *bilateral symmetric interaction* (BSI) game.

An exact potential function for a BSI game is given by (5.12).

$$V(a)=\sum_{i \in N}\sum_{j=1}^{i-1} w_{ij}(a_i,a_j)-\sum_{i \in N} S_i(a_i) \tag{5.12}$$

Again, it is relatively straight-forward to demonstrate that this is an exact potential as $u_i(a_i,a_{-i})-u_i(b_i,a_{-i})=\sum_{j \in N} w_{ij}(a_i,a_j)-\sum_{j \in N} w_{ij}(b_i,a_j)-S_i(a_i)+S_i(b_i)$ and $V(a_i,a_{-i})-V(b_i,a_{-i})=\sum_{j \in N} w_{ij}(a_i,a_j)-\sum_{j \in N} w_{ij}(b_i,a_j)-S_i(a_i)+S_i(b_i)$.

5.2.1.2.7 Multilateral Symmetric Interaction Games (*)

Given a game where all players have utility functions that can be characterized as (5.13)

$$u_i(a) = \sum_{\{S \in 2^N : i \in S\}} w_{S,i}(a_S) + D_i(a_{-i}) \tag{5.13}$$

where $D_i : A_{-i} \rightarrow \mathbb{R}$, S is a particular subset of N to which i belongs, $a_S \in \times_{i \in S} A_i$, $w_{S,i} : A_S \rightarrow \mathbb{R}$, $A_S = \times_{k \in S} A_k$, is a function that assigns a real number to every possible action vectors in A_S, and it is assumed that $w_{S,i}(a_S) = w_{S,j}(a_S) \forall i, j \in S$. Then the game is said to be a *multilateral symmetric interaction* (MSI) game. An expression for the exact potential function for a MSI game is given in (5.14).

$$V(a) = \sum_{S \in 2^N} w_S(a_S) \tag{5.14}$$

Equation (5.14) can be shown to be an exact potential for the MSI game by applying Definition 5.1 and noting that that $\forall i \in N, \forall a \in A$,

$$V(a_i, a_{-i}) - V(b_i, a_{-i}) = \sum_{\{S \in 2^N : i \in S\}} w_{S,i}(a_i, a_{-i}) - \sum_{\{S \in 2^N : i \in S\}} w_{S,i}(b_i, a_{-i})$$

$$u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) = \sum_{\{S \in 2^N : i \in S\}} w_{S,i}(a_i, a_{-i}) - \sum_{\{S \in 2^N : i \in S\}} w_{S,i}(b_i, a_{-i}).$$

The following are some interesting relations between the previously discussed games and the MSI game:

- A coordination game is a MSI game where there is only one subset, S, for which $v_S \neq 0$, and this $S = N$ and $D_i(a) = 0 \forall i \in N$.
- A dummy game is a MSI game where $w_S = 0 \forall S \in 2^N$.
- A coordination-dummy game is the combination of the preceding conditions.
- A self-motivated game is a MSI game where the only coalitions for which $v_S \neq 0$ are those S such that $|S| = 1$.

A BSI game is a MSI game where the only coalitions for which $v_S \neq 0$ are those S such that $|S| = 1$ or $|S| = 2$.

Theorem 5.4: *MSI and Exact Potential Game Equivalence* (*)
Γ is an exact potential game if and only if Γ is a MSI game.
Proof:
$\Rightarrow$(5.14) provides sufficiency.
$\Leftarrow$Necessity is established by leveraging the fact that all coordination-dummy games have a MSI game representation. Thus as all exact potential games must have a coordination-dummy game representation, then they must also have a MSI game representation.

These exact potential game forms, the conditions on the utility functions, and the exact potential are summarized in Table 5.5.

Table 5.5 Common Exact Potential Game Forms

Game	Utility Function Form	Potential Function
Coordination Game	$u_i(a) = C(a)$	$V(a) = C(a)$
Dummy Game	$u_i(a) = D_i(a_{-i})$	$V(a) = c, c \in \mathbb{R}$
Coordination-Dummy Game	$u_i(a) = C(a) + D_i(a_{-i})$	$V(a) = C(a)$
Self-Motivated Game	$u_i(a) = S_i(a_i)$	$V(a) = \sum_{i \in N} S_i(a_i)$
Bilateral Symmetric Interaction (BSI) Game	$u_i(a) = \sum_{j \in N \setminus \{i\}} w_{ij}(a_i, a_j) - S_i(a_i)$ where $w_{ij}(a_i, a_j) = w_{ji}(a_j, a_i)$	$V(a) = \sum_{i \in N} \sum_{j=1}^{i-1} w_{ij}(a_i, a_j) - \sum_{i \in N} S_i(a_i)$
Multilateral Symmetric Interaction (MSI) Game	$u_i(a) = \sum_{\{S \in 2^N : i \in S\}} w_{S,i}(a_S) + D_i(a_{-i})$ where $w_{S,i}(a_S) = w_{S,j}(a_S) \forall i, j \in S$	$V(a) = \sum_{S \in 2^N} w_S(a_S)$

Example 5.5: A Bilateral Symmetric Interaction (BSI) Interference Avoidance Game [Neel_06a]

Consider a network with a frequency reuse scheme such that cross cluster interference is negligible. Each cluster is power controlled so that received power at the cluster head for all radios is constant. However, each radio that is communicating with the cluster head is also attempting to minimize the interference its signal experiences at the receiver by adapting its waveform.

We can model this as a myopic repeated game with the normal form stage game modeled as follows. Each cognitive radio in a cluster is a player, the actions for each radio are its available waveforms, and utility functions given as (5.15) where $r(a_j, a_k)$ is the statistical correlation between waveforms a_j and a_k with the assumption that $r(a_j, a_k) = r(a_k, a_j)$.

$$u_j(a) = -\sum_{k \in N \setminus j} r(a_j, a_k) \quad (5.15)$$

Examining Table 5.5, we can see that (5.15) satisfies the conditions for a BSI game where $S_j(a) = 0 \ \forall j \in N$. Using Table 5.5, we then know that (5.16) is an exact potential function for this game.

$$V(a) = \sum_{i \in N} \sum_{j=1}^{i-1} r(a_i, a_j) \quad (5.16)$$

5.2.1.3 Common Exact Potential Games

Many games frequently encountered in economics can be shown to be exact potential games including the Prisoners' Dilemma, the Cournot Duopoly, and the congestion game discussed in the following.

5.2.1.3.1 Prisoners' Dilemma

Recall that the prisoners' dilemma can be abstractly defined using the 2×2 symmetric game matrix shown in Figure 5.5 where $y < z < w < x$.

Γ	*A*	*B*
a	(*w*, *w*)	(*x*, *y*)
b	(*y*, *x*)	(*z*, *z*)

Figure 5.5 Prisoners' Dilemma Game Matrix

This game can be expressed as a coordination dummy game as shown in Figure 5.6. Note that the exact potential function for this game is given by $C(\cdot)$.

$C(\cdot)$	*A*	*B*
A	$x-z+w-y$	$x-z$
B	$x-z$	0

$D(\cdot)$	*A*	*B*
a	$(z-x+y,\ z-x+y)$	$(z,\ z-x+y)$
b	$(z-x+y,\ z)$	$(z,\ z)$

Figure 5.6: Coordination-Dummy Game Representation of a Prisoners' Dilemma

Note that the Cognitive Radios' Dilemma satisfies these conditions and is thus an exact potential game. Also this formulation is not dependent on the relationships between *x*, *y*, *w*, and *z*, this same formulation holds for any 2×2 symmetric game.

Of interest, [Ui_00] shows that the Prisoners' Dilemma is also a BSI game. The original game can be expressed using the bilateral interaction functions and self-interested functions shown in Figure 5.7 where utility functions are formed as in (5.11).

$w_{ij}(\cdot)$	A	B
a	W	x
b	X	$z+x-y$

$S(\cdot)$	A	B
A	(0, 0)	$(0, x-y)$
b	$(x-y, 0)$	$(x-y, x-y)$

Figure 5.7: BSI Representation of Prisoners' Dilemma.

When (5.12) is applied to find the potential function, a function similar to the one found using the coordination dummy-game method is found. Specifically, the BSI exact potential function is offset by the scalar $z+x-y$. In general, every exact potential game has an infinite number of exact potential functions. However, the difference between any two exact potential functions for this game is always a scalar.

5.2.1.3.2 Cournot Oligopoly

Recall that in a Cournot oligopoly, there are a set of firms which compete in a commodity market by adjusting their production levels in an attempt to maximize their profit. The market is assumed to have an inverse demand function and follow a fixed per-unit cost function for production costs. This game can be modeled as a normal form game with the following components:

- A finite set of players, $N = \{1,2\}$
- $A_1 = A_2 = \left[0, a^{\max}\right]$
- $u_i(a) = a_i\left(M - \sum_{k\in N} a_k\right) - ca_i \; \forall i \in N$ where M is the total market demand.

Following the approach for a Cournot duopoly shown in [Ui_00], we can rewrite the utility functions to yield the following equivalent equation for an oligopoly: $u_i(a) = -\sum_{k\in N\setminus i} a_i a_k + \left(Ma_i - a_i^2 - ca_i\right) \; \forall i \in N$. With this rearrangement, this game can be immediately recognized as a BSI game with $w_{ij}(a) = -a_i a_j$ and

$S_i(a_i) = -(Ma_i - a_i^2 - ca_i)$. Applying (5.12) to this rewritten utility function yields the following exact potential function.

$$V(a) = -\sum_{i\in N}\sum_{k>i}^{n} a_i a_k - \sum_{i\in N}\left(Ma_i - a_i^2 - ca_i\right) \quad (5.17)$$

Note that the action tuple, $a^* = \left(\frac{M-c}{n+1},\ldots,\frac{M-c}{n+1}\right)$ maximizes (5.17) and is also the traditional NE solution of the Cournot duopoly. As we show in Section 5.4, the maximizers of a potential function are always NEs. Finally it is interesting to note that we can also verify that the Cournot oligopoly as formulated is an exact potential game by applying the second order conditions of (5.5) to yield:

$$\frac{\partial^2 u_i(a)}{\partial a_i \partial a_j} = \frac{\partial^2 u_j(a)}{\partial a_j \partial a_i} = -1 \; \forall i, j \in N, a \in A.$$

As it satisfies these basic modeling properties, the Bandwidth Selection game of Example 4.9 is also an exact potential game.

5.2.1.3.3 Congestion Games

The congestion model has the following set of components:

- a finite set of actors (players), $N = \{1,2,\ldots,n\}$,
- a set of facilities, $F=\{1,2,\ldots,g\}$
- a set of payoffs, $c_f(k)$ where $f \in F$ and k is the number of users of facility f.

The congestion game for this model is a strategic form game defined as follows. The player set, N, is the set of actors. The action set, A_i, for each player is the elements of the power set of F (that is each player may choose any subset of facility set, F). And the utility function for each player is the sum of the payoffs for each facility that the player chooses to use, i.e., $u_i(a) = \sum_{f\in a_i} c_f\left(s_f(a)\right)$ where $s_f(a) = \#\{i \in N : f \in a_i\}$ (i.e., the number of players who chose to use facility f).

[Monderer_96] shows that the congestion game is an exact potential game by introducing the potential function:

$$V(a)=\sum_{f\in\bigcup_{i=1}^{n}a_i}\left(\sum_{k=1}^{s_f(a)}c_f(k)\right) \tag{5.18}$$

To show that (5.18) is an exact potential function for the congestion game, consider the following effects that can happen to a particular facility, *f*, when player *i* changes its facility allocation from a_i to b_i:

- $s_f(a)$ can remain the same
- $s_f(a)$ can increase by one
- $s_f(a)$ can decrease by one.

Should $s_f(a)$ be unchanged, then $c_f(s_f(a))$ remains unchanged and there is no change in $u_i(a)$ or $V(a)$. Should player *i* add facility *f*, then $s_f(a)$ increases by one and player *i*'s utility increases by $c_f(s_f(a)+1)$. For the potential function, $\sum_{k=1}^{s_f(a)}c_f(k)$ goes to $\sum_{k=1}^{s_f(a)+1}c_f(k)$ and thus also increases by $c_f(s_f(a)+1)$. If player *i* removes facility *f*, then $s_f(a)$ decreases by one and player *i*'s utility decreases by $c_f(s_f(a))$. For the potential function, $\sum_{k=1}^{s_f(a)}c_f(k)$ goes to $\sum_{k=1}^{s_f(a)}c_f(k)-c_f(s_f(a))$ and thus also decreases by $c_f(s_f(a))$. As both u_i and *V* are sums of the real valuations of the sets of facilities and any change in utilization of a particular facility produces the same change for both u_i and *V*, given allocations a_{-i}, for any unilateral change of allocation of player *i* from a_i to b_i yields the following relationship: $u_i(b_i,a_{-i})-u_i(a_i,a_{-i})=V(b_i,a_{-i})-V(a_i,a_{-i})$. Thus *V* is an exact potential function for the congestion game.

Example 5.6: Distributed Channel Assignment

Consider a collection of radios, each of which are choosing a set of up to two channels to operate on while attempting to maximize their throughput. With a single radio occupying the channel, radio's link can achieve a throughput of 1 Mbps; with two radios in a channel interfering with each other, each radios' link achieves a throughput of 400 kbps; with three 100 kbps; and with four or more radios 0 kbps

This scenario can be modeled as a normal form game as follows.

- The radios as the players, $N = \{1,2,\ldots,n\}$.
- A set of channels, $C=\{1,2,\ldots,c\}$
- Action sets for each player, A_i, which is all subsets of the power set of C that contain two or fewer elements.
- Utility functions of the form $u_i(a) = \sum_{c \in a_i} t_c(s_c(a))$ where $s_c(a) = \#\{i \in N : c \in a_i\}$ (i.e., the number of players who chose to use channel c) and t_c is given by $t_c(0)=0$, . $t_c(1)=1$, $t_c(2)=0.4$, $t_c(3)=0.1$, and $t_c(k)=0, k>3$.

Though with cognitive radios and channels rather than people and facilities (or clubs), this game is also a congestion game. Accordingly, this game is an exact potential game with an exact potential function given by (5.19).

$$V(a) = \sum_{c \in C} \left(\sum_{k=1}^{s_c(a)} t_c(k) \right) \tag{5.19}$$

5.2.2 Ordinal Potential Game Identification

Unlike exact potential games, there are not as many well known game forms for verifying that a game is an ordinal potential game. A common proposed method is to show that a particular game is a generalized ordinal potential game with some special properties that make it an ordinal potential game. Another approach developed by the author is to find a series of ordinal transformations, or better response transformations, such that the resulting game is an exact potential game. The following examines these two approaches.

5.2.2.1 Ordinal Potential Games as Special Generalized Ordinal Potential Games

Two papers – [Monderer_96] and [Voorneveld_97a] – have introduced techniques by which an ordinal potential game can be identified. In [Monderer_96], the following condition is given for an ordinal potential game.

Theorem 5.5: *Ordinal Potential Game Existence*
Suppose a finite game $\Gamma = \langle N, A, \{u_i\} \rangle$ has FIP. If for all $a_{-i} \in A_{-i}$ and all $i \in N$, $u_i(a_i, a_{-i}) \neq u_i(b_i, a_{-i}) \forall a_i, b_i \in A_i$ then Γ is an ordinal potential game.
Proof : Since the game has FIP, then it has a generalized ordinal potential, V. However, since $u_i(a_i, a_{-i}) \neq u_i(b_i, a_{-i}) \forall a_i, b_i \in A_i$, $u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \Leftrightarrow V(a_i, a_{-i}) > V(b_i, a_{-i})$.

A similar formulation introduced in [Voorneveld_97a] gives the following pair of conditions for establishing that a finite game Γ is an ordinal potential game:

1. Γ lacks *weak improvement cycles*
2. A is *properly ordered* on the preference relationship, $\prec$.

A discussion of these conditions necessitates the introduction of a number of definitions.

Definition 5.6: *Non-deteriorating Path*
A path, g, is said to be *non-deteriorating* if $u_i(a^{k-1}) \leq u_i(a^k)$ for all $a^k \in g$ where i is the unique deviator at step k.

Definition 5.7: *Weak Improvement Cycle*
A path, g, is said to be a *weak improvement cycle* if g is a cycle, g is non-deteriorating, and for at least one $a^k \in g$, $u_i(a^k) < u_i(a^{k+1})$.

Definition 5.8: *Properly Ordered Set*
A set, X, is said to be properly ordered with order $\prec$ where $\prec$ is an irreflexive and transitive binary relation if there exists a function $F: X \rightarrow \mathbb{R}$ such that $x \prec y \Rightarrow F(x) \prec F(y) \ \forall x, y \in X$.

These definitions allow [Voorneveld_97a] to introduce the following theorem for identifying ordinal potential games.

Theorem 5.6: *Ordinal Potential Games and Weak Improvement Cycles* [Voorneveld_97a]

A normal form game $\Gamma = \langle N, A, \{u_i\} \rangle$ is an ordinal potential game if and only if the following two conditions are satisfied:

1) A has no weak improvement cycles

2) $(A, \prec)$ is properly ordered.

Proof: A proof is given in [Voorneveld_97a].

Note that for countable games, the conditions of Theorem 5.6 can be relaxed to only a lack of weak improvement cycles.

Example 5.7: Ordinal Potential Games and Generalized Ordinal Potential Games

Consider the normal form game Γ_1 shown in Figure 5.8. As we showed in Figure 5.1, Γ_1 is a generalized ordinal potential game and has FIP but is not an ordinal potential game. Note that $u_{col}(a, A) = u_{col}(a, B)$ so the conditions for Theorem 5.5 are not satisfied. Also as γ=((a,A), (b,A), (b,B), (a,B), (a,A)) is a weak improvement cycle, Theorem 5.6 says that Γ_1 cannot be an ordinal potential game either.

Γ_1	*A*	*B*
a	(1,0)	(2,0)
b	(2,0)	(0,1)

Figure 5.8: A Generalized Ordinal Potential Game.

Now consider the normal form game Γ_2 shown in Figure 5.9 which we previously showed to be an ordinal potential game. In Γ_2, which has FIP, $u_i(a_i, a_{-i}) \neq u_i(b_i, a_{-i})$ $\forall a_i, b_i \in A_i$ and there is no weak improvement cycle. So by Theorem 5.5 and Theorem 5.6, Γ_2 is an ordinal potential game.

Γ_2	*A*	*B*
A	(1,-1)	(2,0)
B	(2,0)	(0,1)

Figure 5.9: An Ordinal Potential Game.

5.2.2.2 Equivalence Relationships (*)

Somewhat at odds with our original intention of introducing potential games, applying the techniques in the preceding section requires one to exhaustively search for weak improvement cycles or to first establish that a game has FIP and then verify that there are no action vectors for which a unilaterally deviating player would be indifferent.

This section takes a different approach and introduces a new technique for identifying if a normal form game is an ordinal potential game – the use of *better response equivalencies*. This section defines the concepts required to establish better response equivalence and introduces a number of valuable theorems for identifying ordinal potential games.

> **<u>Definition 5.9</u>**: *Better-response equivalence*
> A game $\Gamma = \langle N, A, \{u_i\} \rangle$ is said to be *better response equivalent* to game $\Gamma' = \langle N, A, \{v_i\} \rangle$ if $\forall i \in N, a \in A$, $u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \Leftrightarrow v_i(a_i, a_{-i}) > v_i(b_i, a_{-i})$.

For notational convenience we indicate that Γ is better response equivalent to Γ' by writing $\Gamma \approx \Gamma'$. We can also define a similar concept - *best response equivalence*.

> **<u>Definition 5.10</u>** *Best-response equivalence* [Morris_02]
> A game $\Gamma = \langle N, A, \{u_i\} \rangle$ is said to be *best response equivalent* to game $\Gamma' = \langle N, A, \{v_i\} \rangle$ if $\forall i \in N, a \in A$, $\underset{a_i \in A_i}{\operatorname{argmax}}\, u_i(a_i, a_{-i}) = \underset{a_i \in A_i}{\operatorname{argmax}}\, v_i(a_i, a_{-i})$.

For notational convenience we will indicate that Γ is best response equivalent to Γ' by writing $\Gamma \underset{\max}{\approx} \Gamma'$. We can also define a similar concept *e-best response equivalence*.

> **<u>Definition 5.11</u>**: *ε-better-response equivalence* (*)
> A game $\Gamma = \langle N, A, \{u_i\} \rangle$ is said to be ε-*better response equivalent* to game $\Gamma' = \langle N, A, \{v_i\} \rangle$ there are $e_1, e_2 > 0$ such that $u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) + e_1 \Leftrightarrow v_i(a_i, a_{-i}) > v_i(b_i, a_{-i}) + e_2$ $\forall i \in N, a \in A$.

For notational convenience we will indicate that Γ is ε-better response equivalent to Γ' by writing $\Gamma \underset{e}{\approx} \Gamma'$.

Theorem 5.7: Better response equivalence of ordinal potential games and coordination games (*)

Given an ordinal potential game, $\Gamma = \langle N, A, \{u_i\} \rangle$, with potential V, then Γ is better response equivalent to the coordination game defined as $\Gamma' = \langle N, A, \{v_i = V\} \rangle$.

Proof: As Γ is an ordinal potential game, $u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \Leftrightarrow V(a_i, a_{-i}) > V(b_i, a_{-i})$. Since $v_i = V$, $u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \Leftrightarrow v_i(a_i, a_{-i}) > v_i(b_i, a_{-i})$ and $\Gamma \approx \Gamma'$.

Theorem 5.7 leads to the following interesting result.

Corollary: (*)

All ordinal potential games are better response equivalent to an exact potential game.

Proof: By Theorem 5.7, all ordinal potential games are better response equivalent to a coordination game. As we showed previously, all coordination (identical-interest) games are also exact potential game.

Using the idea of better response equivalence, a similar corollary can be formulated.

Corollary: (*)

Γ is an ordinal potential game if and only if f there exists a game $\Gamma' = \langle N, A, \{C + D_i\} \rangle$ where $C : A \to \mathbb{R}$ and $D_i : A_{-i} \to \mathbb{R}$ such that $\Gamma \approx \Gamma'$.

This implies that a different technique can be used to identify if a game is an ordinal potential game, specifically showing that the there exists another game which is an exact potential game and better response equivalent to the original game.

Theorem 5.8: Identifying Ordinal Potential Games (*)

A normal form game $\Gamma = \langle N, A, \{u_i\} \rangle$ is an ordinal potential game if and only if it is better response equivalent to an exact potential game.

Proof: By the second corollary above, a Γ is an ordinal potential game if and only if there exists a game $\Gamma' = \langle N, A, \{C + D_i\} \rangle$ where $C : A \to \mathbb{R}$ and $D_i : A_{-i} \to \mathbb{R}$ such that $\Gamma \approx \Gamma'$. By Theorem 5.2, Γ' is an exact potential game. So Γ is an ordinal potential game if and only if Γ is better response equivalent to an exact potential game.

5.3 Special Properties of Potential Games

The following are some valuable properties of potential games which are not directly related to steady-states, optimality, convergence, or noise. Specifically, this section considers the relationship between FIP and potential games, the relationships between AFIP and potential games, the implications of equivalence properties on FIP and AFIP, continuity properties of potential games, the net improvement properties of exact potential games, and the linear space properties of exact potential games.

5.3.1 FIP and Potential Games

When we introduced potential games, we stated without proof that all potential games have FIP. In fact, we strengthen this claim in the following by showing that a finite normal form game has FIP if and only if the game has a generalized ordinal potential function.

Theorem 5.9: *FIP and Generalized Ordinal Potential Games*
All finite generalized ordinal potential games have FIP.
Proof: (Along the lines of a proof given in [Monderer_96]) Suppose $\Gamma = \langle N, A, \{u_i\} \rangle$ is a generalized ordinal potential game with potential V. Now consider any improvement path $g = (a^0, a^1, \ldots)$ in A. Then $u_i(a^{k+1}) > u_i(a^k)$ where i is the unique deviator at step k+1. As Γ is a generalized ordinal potential game, $u_i(a^{k+1}) > u_i(a^k) \Rightarrow V(a^{k+1}) > V(a^k)$. Then $V(a^0) < V(a^1) < \ldots$ and $V(g)$ forms a monotonically increasing sequence. Since A is finite and $V(g)$ is monotonic, g must be finite.

Establishing a relationship in the reverse direction requires the introduction of a few preliminary results and terms. Define $s(a)$ as the set of improvement paths that terminate in a and $\mathcal{L}(s(a))$ as the length of the longest improvement path that terminates in a. Now consider (5.20) as a candidate generalized ordinal potential function for any game with FIP and the following theorem.

$$V(a) = \mathcal{L}(s(a)) \tag{5.20}$$

Theorem 5.10: Generalized Ordinal Potential Games and FIP (*)
All games with FIP are generalized ordinal potential games.
Proof: Equation (5.20) can be verified as a generalized ordinal potential function for any game that has FIP. Consider any $u_i(b_i, a_{-i}) > u_i(a_i, a_{-i})$. Then there exists an improvement path from $u_i(a_i, a_{-i})$ to $u_i(b_i, a_{-i})$. By (5.20) $V(b_i, a_{-i}) \geq V(a_i, a_{-i}) + 1$. Thus $u_i(b_i, a_{-i}) > u_i(a_i, a_{-i}) \Rightarrow V(b_i, a_{-i}) > V(a_i, a_{-i})$ and Definition 5.4 is satisfied.

Theorem 5.10 also appears in [Milchtaich_96] with a different potential function, specifically the *"integer-valued function that assigns to a strategy-tuple a the number of strategy-tuples which are the initial point of an improvement path with the terminal point a is easily seen to be a generalized ordinal potential*." However, for infinite games that have FIP, this potential function can quickly become unevaluatable.

For instance, consider the following slight modification to the infinite game with FIP presented in Chapter 4. Consider the three player game where $A_1 = A_2 = A_3 = [0,1]$ where the utility functions for this game be given by (5.21) and (5.22).

$$u_1(a) = u_2(a) = \begin{cases} 1 & a_i = 0 \\ 0 & otherwise \end{cases} \tag{5.21}$$

$$u_3(a) = \begin{cases} 1 & a = (0,0,0) \\ 0 & otherwise \end{cases} \tag{5.22}$$

However, under the potential function of [Milchtaich_96], $V(1,1,0) = \infty$ and $V(1,1,1) = \infty$ (specifically $\infty + \infty$), yet $u_3(1,1,1) > u_3(1,1,0)$ which would imply that $\infty > \infty$, an uncomfortable proposition. However, this game does have FIP as the longest improvement path has a length of 3.

One might suppose that the potential of (5.20) suffers from the same problem. However, this cannot be the case as the largest value of (5.20) is the length of the longest improvement path in the game. As FIP implies that the length of no improvement path in the game is unbounded, there can be no a for which $\mathcal{L}(s(a))$ is unbounded. Specifically, the largest value that $\mathcal{L}(s(a))$ assumes is always the length of the longest improvement path in the game. For example, in the previous example the largest value that $\mathcal{L}(s(a))$ assumes is 3.

Combining the results of Theorem 5.9 and Theorem 5.10 yields the following.

Theorem 5.11: *Equivalence of Generalized Ordinal Potential Games and FIP*[3]
A finite normal form game has FIP if and only if it has a generalized ordinal potential.
Proof: Sufficiency and necessity are supplied by Theorem 5.9 and Theorem 5.10, respectively.

For our purposes, this is a very significant result as it provides a mechanism for determining if a game has FIP – specifically identifying that a game is an exact or ordinal potential game using the techniques of Section 5.2. This then enables us to apply the convergence criteria and noise properties of Chapter 4 to a broad class of readily identifiable games.

5.3.2 Approximate Finite Improvement Property (AFIP)

For a finite game, the definition of FIP is equivalent to saying that the game lacks improvement cycles. However for games with convex action spaces, this definition is lacking as an infinite sequence of increasingly smaller improvements can be created without encountering an improvement cycle. For instance, consider the following game which we'll call "Zeno's Game" based on Zeno's Dichotomy or Race Course paradox.

Example 5.8: Zeno's Game

In Zeno's game, there are two players, {1,2}. Player 1 chooses to walk a distance d_1 contained in the range [0,1] and player 2 also chooses to walk a distance d_2 contained in the range [0,1]. The player that chooses the shorter distance must pay the other player the number of dollars indicated by the difference in distances, i.e., $u_1(d) = d_2 - d_1$ and $u_2(d) = d_1 - d_2$.

Now consider the following sequence of actions, $\{d^k\}$. For d^0, player 2 always chooses $d_2 = 1$ and player 1 chooses $d_1 = 0$. All other elements in $\{d^k\}$, player 1's choice of distances is equal to its current distance plus half of the remaining distance. This process

[3] While unbounedness for infinite games with FIP is a problem for the assertion in [Milchtaich_96], it is not a problem for finite games as V could never take on values greater than |A|. Thus this assertion has been proven before even though one of the predicate theorems in this proof had not been previously shown.

generates the action sequence $d^k = \left(\frac{2^k - 1}{2^k}, 1 \right)$ where k goes from 0 to ∞. Note that $\{d^k\}$ constitutes an improvement path where player 1 improves its payoff by $1 - \frac{2^k - 1}{2^k}, k \geq 1$. However this improvement path continues to improve for all k making the path an infinite improvement path. Yet this game has an exact potential given by $V(d) = d_1 + d_2$.

Clearly, not every potential game has FIP. However, such an infinite improvement path seems counter-intuitive to our experiences. We all know that Zeno's paradox (on which this game is based) is a false paradox as we have all walked across a room where an equivalent infinite improvement path exists. This is because we don't take infinitesimally small steps - nor could we even if we wanted. Instead, there is a limit to our how small step sizes can be, a concept leveraged in the *e-improvement path*.

Definition 5.12: *e-improvement path*
Given $e > 0$, an *e-improvement path* is a path such that for all $k \geq 1$, $u_i(a^k) > u_i(a^{k-1}) + e$ where i is the unique deviator at step k.

While it may not be possible to generally guarantee that an infinite convex game has finite improvement paths for potential games, it seems reasonable that it would be possible to guarantee when all e-improvement paths are finite. This concept is formalized in the *Approximate Finite Improvement Property* (AFIP) [Monderer_96].

Definition 5.13: *Approximate Finite Improvement Property* (*AFIP*)
A normal form game, Γ, is said to have to have the *approximate finite improvement property* if for every ε>0 there exists an $L(e) \in \mathbb{N}$ such that the length of all ε-improvement paths in Γ are less than or equal to L.

In effect, AFIP states that all sequences of selfish deviations that improve the deviator's payoff by at least some arbitrarily, but minimally small amount must be finite. Of course, FIP implies AFIP.

Theorem 5.12: *FIP and AFIP* (*)
All games with FIP also have AFIP.
Proof: By definition, all ε-improvement paths are also improvement paths. So if all improvement paths are finite, all ε-improvement paths must also be finite.

An obvious condition for a game to have AFIP is a bounded action space. This condition is not necessary, however – just consider the example infinite game with FIP but with the action sets extending to ∞. Further while boundedness proves useful, we cannot readily establish that bounded generalized ordinal potential games or ordinal potential games have AFIP as the action space is no longer finite. However, it can be shown that weighted potential games with bounded potential functions have AFIP.[4]

Theorem 5.13: *Weighted Potential Games and AFIP* (*)[4]
Every weighted potential game with a bounded potential has AFIP.
Proof: Consider a bounded infinite weighted potential game, $\Gamma = \langle N, A, \{u_i\}\rangle$ with bounded potential function V and weights $\{a_i\}$ with $a_{\max} = \max_{i\in N}\{a_i\}$. Then given $e > 0$ and any ε-improvement path in Γ, γ, $V(\gamma)$ forms a bounded monotonically increasing sequence with minimum step size $e / a_{\max}$. Thus $V(\gamma)$ must be finite and γ must be finite.

We are able to establish a similar result for bounded generalized ε-ordinal potential games in Theorem 5.14.

Theorem 5.14: *Generalized e-potential game and AFIP* (*)
Every generalized ε-potential game with a bounded potential has AFIP.
Proof: Consider a generalized ε- ordinal potential games, $\Gamma = \langle N, A, \{u_i\}\rangle$ with potential function bounded with $|V(a)| \leq B < \infty$. Given $e_1 > 0$, for every ε_1-improvement path, $\gamma=\{a^0, a^1, \ldots, a^n,\ldots\}$ in Γ, there is an $\varepsilon_2 > 0$ such that $V(a^0)+ \varepsilon_2 < V(a^1)$, $V(a^1)+ \varepsilon_2 < V(a^2)$, $V(a^k)+ \varepsilon_2 < V(a^{k+1})$ and $V(a^0)+ k\varepsilon_2 < V(a^k)$. Now suppose γ is infinite. Then for $k = \lceil 2B/e_2 \rceil$, $V(a^0)+ 2B < V(a^k)$. But $|V(a)| \leq B < \infty$ so $V(a^0)+ 2B \geq V(a^k)$. So γ cannot be longer than $k = \lceil 2B/e_2 \rceil$ and thus γ cannot be infinite. So every generalized ε-potential game with a bounded potential has AFIP.

We can also establish that every game with AFIP is a generalized ε-ordinal potential game by identifying a generalized ε-ordinal potential function as follows. Given $e > 0$,

[4] This may or may not be a new result due to ambiguous language in [Monderer_96].

define $s_e(a)$ as the set of e-improvement paths that terminate in a. Define $\mathcal{L}(s_e(a))$ as the length of the longest e-improvement path that terminates in a. Then the following function is a generalized ε-potential function for any game with AFIP.

$$V(a) = \mathcal{L}(s_e(a)) \tag{5.23}$$

(5.23) can be verified as a generalized ε-ordinal potential function for any game that has AFIP. Suppose $u_i(b_i, a_{-i}) > u_i(a_i, a_{-i}) + e$. Then there exists an ε-improvement path from $u_i(a_i, a_{-i})$ to $u_i(b_i, a_{-i})$ so $V(b_i, a_{-i}) \geq V(a_i, a_{-i}) + 1$. So $u_i(b_i, a_{-i}) > u_i(a_i, a_{-i}) + e$ implies $V(b_i, a_{-i}) > V(a_i, a_{-i})$ and Definition 5.5 is satisfied. This result allows us to introduce the following theorem.

Theorem 5.15: *AFIP and Generalized e-Potential Games* (*)
A game has AFIP only if it is a generalized ε-potential game.
Proof: Equation (5.23) provides the necessary potential function to satisfy Definition 5.5.

5.3.3 Improvement Path Implications of Equivalence Properties

In Section 5.2.2.2, we introduced a number of equivalence relationships which we can exploit to establish a number of useful analytic results for NEs, FIP, and AFIP.

Theorem 5.16: *Best Response Equivalence and NE*
If $\Gamma \underset{\max}{\approx} \Gamma'$, then the NE of Γ, if any exist, are coincident with the NE of Γ'
Proof: Suppose a^* is a NE of Γ, $\Gamma = \langle N, A, \{u_i\} \rangle$, and $\Gamma' = \langle N, A, \{v_i\} \rangle$ Then $a^* \in \underset{a_i \in A_i}{\operatorname{argmax}}\, u_i(a_i, a_{-i}) \, \forall i \in N$. Since $\underset{a_i \in A_i}{\operatorname{argmax}}\, u_i(a_i, a_{-i}) = \underset{a_i \in A_i}{\operatorname{argmax}}\, v_i(a_i, a_{-i})$, $a^* \in \underset{a_i \in A_i}{\operatorname{argmax}}\, v_i(a_i, a_{-i}) \;\; \forall i \in N$. Thus a^* is a NE for Γ' as well. Similar logic holds in the reverse direction.

As every game that better response equivalence implies best response equivalence, a similar result can be found for better response equivalent games.

Corollary: *Better Response Equivalence and NE* (*)
If $\Gamma \approx \Gamma'$, then the Nash equilibria of Γ, if any exist, are coincident with the Nash equilibria of Γ'.
Proof: If two games are better response equivalent, then they are also best response equivalent.

So if the analysis of a game Γ proves difficult, we can instead analyze any other game Γ' where $\Gamma' \approx \Gamma$ or $\Gamma' \underset{\max}{\approx} \Gamma$ to solve for the Nash equilibria of Γ. Further, a better response transformation preserves FIP.

Theorem 5.17: Better Response Equivalence and FIP (*)
Given $\Gamma = \langle N, A, \{u_i\} \rangle$ and $\Gamma' = \langle N, A, \{v_i\} \rangle$ with $\Gamma \approx \Gamma'$ and Γ finite, then if Γ has FIP, then so does Γ'.
Proof: By , Γ has a generalized ordinal potential, V. Since for all $a \in A$, $u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \Leftrightarrow v_i(a_i, a_{-i}) > v_i(b_i, a_{-i})$ and $u_i(b_i, a_{-i}) - u_i(a_i, a_{-i}) > 0 \Rightarrow$, $V(b_i, a_{-i}) - V(a_i, a_{-i}) > 0$ then $v_i(b_i, a_{-i}) - v_i(a_i, a_{-i}) > 0 \Rightarrow V(b_i, a_{-i}) - V(a_i, a_{-i}) > 0$ and Γ' is a generalized ordinal potential game. By Theorem 5.10 again, Γ' must have FIP.

We can establish a similar relationship for better response transformations and ordinal potential games.

Theorem 5.18: *Ordinal Potential Games and Better Response Equivalence* (*)
If Γ is an ordinal potential game and $\Gamma \approx \Gamma'$, then Γ' is also an ordinal potential game.
Proof: Suppose $\Gamma = \langle N, A, \{u_i\} \rangle$ with ordinal potential, V, and $\Gamma' = \langle N, A, \{v_i\} \rangle$. We know that $u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \Leftrightarrow v_i(a_i, a_{-i}) > v_i(b_i, a_{-i}) \quad \forall i \in N, a \in A$ and $u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \Leftrightarrow V(a_i, a_{-i}) > V(b_i, a_{-i})$. By the transitivity of $>$ we thus have $v_i(a_i, a_{-i}) > v_i(b_i, a_{-i}) \Leftrightarrow V(a_i, a_{-i}) > V(b_i, a_{-i})$. So V is also an ordinal potential for Γ'.

Finally, we can use the *e*-better response equivalence to establish AFIP.

Theorem 5.19: *AFIP Equivalence* (*)
If $\Gamma \underset{e}{\approx} \Gamma'$, then if Γ has AFIP, then so does Γ'.
Proof: Suppose $\Gamma = \langle N, A, \{u_i\} \rangle$ and $\Gamma' = \langle N, A, \{v_i\} \rangle$. By Theorem 5.15, Γ has AFIP iff Γ is a generalized ε-potential game with potential function V. Since
$u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) + e_1 \Leftrightarrow v_i(a_i, a_{-i}) > v_i(b_i, a_{-i}) + e_2 \ \forall i \in N, a \in A$ if
$u_i(b_i, a_{-i}) - u_i(a_i, a_{-i}) > e_1 \Rightarrow V(b_i, a_{-i}) - V(a_i, a_{-i}) > e_3$ then
$v_i(b_i, a_{-i}) - v_i(a_i, a_{-i}) > e_2 \Rightarrow V(b_i, a_{-i}) - V(a_i, a_{-i}) > e_3$.

Example 5.9: Identifying An Ordinal Potential Game

Consider the game $\Gamma = \langle N, A, \{u_i\} \rangle$ where $N = \{1,2,\ldots,n\}$, $A_i = [0,1] \quad \forall i \in N$, $u_i(a) = -\left| a_i - \frac{1}{2|N|} \sum_{k \in N \setminus i} a_k \right|$, $\forall i \in N$. The second derivative condition can not be applied as the first derivative is undefined for a wide range of values. However, consider the better response equivalent game with utility functions defined as $v_i(a) = -\left(a_i - \frac{1}{2|N|} \sum_{k \in N \setminus i} a_k \right)^2$. Evaluating second order derivatives, we get the following $\frac{\partial^2 v_i(a)}{\partial a_i \partial a_j} = \frac{\partial^2 v_j(a)}{\partial a_i \partial a_j} = -\frac{1}{|N|} \forall i \neq j$. Thus the transformed game is an exact potential game, and the original game is an ordinal potential game. Further any exact (or ordinal) potential function for Γ' is also an ordinal potential for Γ. By applying this simple transformation, we are now able to apply potential game analysis techniques to the original game.

5.3.4 Continuity Properties of Potential Games

In subsequent sections, the continuity of a game's potential function is critical to establishing the steady-state, convergence, and stability properties for infinite potential games. Beyond directly evaluating continuity for each potential function, it is also possible to infer potential function continuity from utility function properties.

Before discussing the continuity properties of potential games, we need to introduce a general result related to the continuity of the sum of functions. Recall that a function, $f : A \to \mathbb{R}$ is said to be *continuous* at $a \in A$ where A is a metric space if for every $e > 0$ there is a $d > 0$ such that $\|a' - a\| < d$ implies $\|f(a' - a)\| < e$.

Theorem 5.20: *Continuity of a Sum of Functions*

Given $h = f + g : A \to \mathbb{R}$ where g is continuous, h is continuous if and only if f is continuous.

Proof:

(⇒) Proceeding with a proof by construction. For f, g continuous and for all $a \in A$, given $h > 0$ there are d_f, d_g such that $\|a, a'\| < d_f \Rightarrow \|f(a), f(a')\| < h$ and $\|a, a'\| < d_g \Rightarrow \|g(a), g(a')\| < h$. If we set $h \leq e/2$, then $d_h = \min(d_f, d_g)$ implies $\|a, a'\| < d_h \Rightarrow \|h(a), h(a')\| < e$.

(⇐) Suppose at $a \in A$, f is discontinuous and h is continuous. As as h is continuous for every ε>0 there is some d_h>0 such that $\|a, a'\| < d_h \Rightarrow \|h(a), h(a')\| < e$ or $\|a, a'\| < d_h \Rightarrow \|f + g(a), f + g(a')\| < e$. But this implies that $\|a, a'\| < d_h \Rightarrow \|f(a), f(a')\| < e$ as $\|f + g(a), f + g(a')\| \geq \|f(a), f(a')\|$. Thus for every ε>0, $d_f = d_h$ would supply a sufficient d_f that $\|a, a'\| < d_f \Rightarrow \|f(a), f(a')\| < e$ contradicting the assumption that f is discontinuous at a. Therefore h cannot be continuous.

Theorem 5.20 proves very useful when considering when an exact or weighted potential game will have a continuous potential function. Also note that continuity of u_i alone does not imply continuity of V as D_i and C could have equal and opposite jump discontinuities.

Theorem 5.21: *Continuity of Weighted Potential Function* (*)

Given an weighted potential game, $\Gamma = \langle N, A, \{u_i\} \rangle$ with weights $\{a_i\}$ and potential function $V : A \to \mathbb{R}$ where A is a compact metric space, if $u_i : A \to \mathbb{R}$ is continuous in a and $D_i(a_{-i})$ is continuous in a $\forall i \in N$, then V is uniformly continuous in a for all $i \in N$.

Proof: Recall that by Theorem 5.3 that Γ is a weighted potential game if and only if there exist functions $C : A \to \mathbb{R}$ and $D_i : A_{-i} \to \mathbb{R}$ and scalars $a_i \in \mathbb{R}$ such that $u_i(a) = a_i C(a) + D_i(a_{-i})$ $\forall i \in N, a \in A$. By Theorem 5.20, the continuity of D_i and u_i implies that C is continuous as well. As any potential function for a weighted potential game must be of the form $V(a) = a_V C(a) + k$ and continuity is preserved for linear transformations, $V(a)$ is continuous as well. As V is continuous, V is also uniformly continuous in a.

Note that the same conditions imply that an exact potential function is uniformly continuous as an exact potential game is a weighted potential game. Unfortunately, not all ordinal potential games are continuous even with continuous utility functions and a compact action space. For instance, Theorem 4.1 in [Voorneveld_97b] gives the following normal form game which is shown to be an ordinal potential game, but for which no continuous potential function can exist.

Example 5.10: An Ordinal Potential Game without a Continuous Potential Function [Voorneveld_97b]

Consider the normal form game, $\Gamma = \langle N, A, \{u_i\} \rangle$, with $N = \{1,2\}$, $A_1 = A_2 = [0,1]$ and payoff functions defined as

$$u_1(a_1,a_2) = \begin{cases} 0 & (a_1,a_2) = (0,0) \\ \dfrac{a_1 a_2^6}{\left(a_1^2 + a_2^2\right)^3} & otherwise \end{cases}$$

$$u_2(a_1,a_2) = \begin{cases} 0 & (a_1,a_2) = (0,0) \\ \dfrac{a_1^6 a_2}{\left(a_1^2 + a_2^2\right)^3} & otherwise \end{cases}.$$

Then an ordinal potential function for this game is given by

$$V(a) = \begin{cases} 0 & (a_1,a_2) = (0,0) \\ \dfrac{a_1 a_2}{\left(a_1^2 + a_2^2\right)^3} & otherwise \end{cases}.$$

The sequence $g = \left\{\left(2^{-\lceil n/2 \rceil}, 2^{-\lfloor n/2 \rfloor}\right)\right\}$ forms an infinite improvement path. However, as shown in [Voorneveld_97b]. $V(\gamma)$ is an unbounded monotonically increasing sequence and thus has no maximum. As A is compact and V has no maximum, V cannot be continuous. [Voorneveld_97b] goes on to show that every ordinal potential game for Γ must be strictly increasing on γ and thus does not have a maximum. Accordingly there is no continuous ordinal potential function for Γ.

5.3.5 Net Improvement Properties of Exact Potential Games

Given a finite normal form game with utility functions, $\{u_i\}$and a finite path $g = \left(a^0, a^1, \ldots, a^m\right)$, we define the *net improvement* of the path, $I(g, \{u_i\})$ as in (5.24)

$$I\left(g, \{u_i\}\right) = \sum_{k=1}^{m} \left[u_{i_k}\left(a^k\right) - u_{i_k}\left(a^{k-1}\right) \right] \tag{5.24}$$

where i_k is the unique deviator at step k.

Example 5.11: Example Calculation of Net Improvement

Consider the finite normal form game shown in Figure 5.10 and the path, $\gamma = ((a,A), (a,B), (b,B), (b,A), (a,A))$. We calculate the net improvement as $I(\gamma, \{u_i\}) = 2 + 1 - 1 - 2 = 0$.

Γ	*A*	*B*
a	(3,3)	(0,5)
b	(5,0)	γ (1,1)

Figure 5.10 Exact Potential Game Γ

The fact that $I(\gamma, u)=0$ in this example is not by chance. In Example 5.1, it was shown that this game is an exact potential game. For all exact potential games, there is a special property for finite closed cycles made explicit in Theorem 5.22.

Theorem 5.22: *Exact Potential Games and Net Improvement on a Cycle* [Monderer_96]

Let Γ be a finite normal form game. Then the following claims are equivalent:

(1) Γ is an exact potential game.
(2) $I(\gamma,u) = 0$ for every finite closed cycle γ.
(3) $I(\gamma,u) = 0$ for every finite simple closed cycle γ.
(4) $I(\gamma,u) = 0$ for every finite simple closed cycle γ of length 4.

Proof: The proof of this theorem is omitted here due to its length. However, a full proof is provided in Appendix *A* of [Monderer_96].

Theorem 5.22 also forms the theoretical basis of the following algorithm for identifying exact potential games.

Motivated by Theorem 5.2 (coordination dummy games and potential games) and assured by Theorem 5.22, we can establish the following algorithm which verifies that a game is an exact potential game by attempting to find its constituent coordination and dummy games. When applied to a finite exact potential game, the coordination-dummy algorithm yields the game's constituent coordination and dummy functions. If the coordination-dummy game algorithm is applied to a game that is not finite exact potential game, then the algorithm fails by giving different values for the same action tuple or by continuing indefinitely. .

Algorithm 5.1: Coordination-Dummy Game Algorithm

Coordination Function Identification

(1) Pick any initial action tuple, a^*. Assign $C(a^*) = 0$.

(2) Define the set A^* as the set of all profitable unilateral deviations from a^* contained in A.

(3) For all $a \in A^*$, assign $C(a) = C(a^*) + u_i(a) - u_i(a^*)$ where i is the unique deviator from a^* to a.

(4) Pick any $a \in A^*$. Define $a^* = a$.

(5) Repeat steps 2-4 until $C(a)$ is defined for all $a \in A$.

If at any step two different valuations of $C(a)$ are found, then the algorithm fails and the game is not an exact potential game.

Dummy Function Identification

Assuming the coordination function was successfully identified, then the dummy functions can be found as $d_i(a) = u_i(a) - C(a)$.

5.3.6 Linear Space of Exact Potential Games

Interestingly, [Fachini_97] shows that the set of all exact potential games with player set N and action space A form a linear (vector) space. This means that when we take linear combinations of exact potential games with a shared player set and common action space, the result is also an exact potential game.

Definition 5.14: *Linear Space*

Given a set X, X is said to be a *linear space* if for every $x, y, z \in X$ and every $a, b \in \mathbb{R}$ it satisfies the following ten (10) properties:

(1) Closure under addition, $x + y \in X$

(2) Closure under scalar multiplication, i.e., $ax \in X$

(3) Commutativity, i.e., $x + y = y + x$

(4) Additive Associativity, i.e., $x + y = y + x$

(5) Additive Identity, i.e., there is some $0 \in X$ such that if $x \in X$, $0 + x = X$.

(6) Additive Inverse, i.e., for every $x \in X$, there is some $-x \in X$ such that $x + (-x) = 0$.

(7) Associativity of Scalar Multiplication, i.e., $a(bx) = (ab)x$

(8) Distributivity of Scalar Sums, i.e., $(a + b)x = ax + bx$

(9) Distributivity of Vector Sums, i.e., $a(x + y) = ax + ay$

(10) Scalar Multiplicative Identity, i.e., $1x=x$.

To prove that the set of exact potential games form a linear space (or more accurately that subsets of exact potential games form linear spaces), we must first define the set of exact potential games and the addition and scalar multiplication operations for exact potential games.

Following the notation in [Fachini_97], we define $\Gamma^{N,A}$ as the family of normal form games with player set, $N=\{1,\ldots,n\}$, action space $A=\times_{i\in N} A_i$, and utility functions, $u_i : A\rightarrow\mathbb{R}$ $i\in N$, such that for each $\Gamma\in\Gamma^{N,A}$, Γ is an exact potential game (implying possibly different potential functions). We'll define the addition of two games, $\Gamma_1,\Gamma_2\in\Gamma^{N,A}$, $\Gamma_1=\langle N, A,\{u_i\}\rangle$, $\Gamma_2=\langle N, A,\{v_i\}\rangle$ as $\Gamma_1+\Gamma_2=\Gamma_3=\langle N, A,\{u_i+v_i\}\rangle$. We'll further define the scalar multiplication of game Γ_1 by $a\in\mathbb{R}$ as $a\Gamma_1=\langle N, A,\{a u_i\}\rangle$. These conventions permit the introduction of the following theorem.

Theorem 5.23: *Linear Space of Exact Potential Games* [Fachini_97]
$\Gamma^{N,A}$ forms a linear space
Proof: A proof of this result is given in [Fachini_97]. However, some key aspects of this proof are repeated in the following. An additive identity element is given by the game $\Gamma=\langle N, A,\{0\}\rangle$ which has exact potential function $V(a)=0$. Given exact potential games, Γ_1, $\Gamma_2\in\Gamma^{N,A}$ with potential functions V_1 and V_2, and scalars $a_1,a_2\in\mathbb{R}$, $\Gamma_3=a_1\Gamma_1+a_2\Gamma_2$, then Γ_3 is an exact potential game with potential $V_3=\alpha_1 V_1+\alpha_2 V_2$.

Applications of Theorem 5.23 are given in Example 5.12 and Example 5.14. In general, the knowledge that we can scale, add constants, and add together exact potential games and still get a potential game is a valuable insight for cognitive radio analysis and design. For instance once a cognitive radio designer shows that his cognitive radio network forms an exact potential game, any number of individual cost functions can be additively introduced to the radios utility function while preserving the implication that the network is an exact potential game. We return to this topic in Section 5.5 when we discuss the desirability and optimality of the steady-states of potential games.

Example 5.12: Linear Combination of Exact Potential Games

Consider the finite normal form games shown in Figure 5.11. These games are exact potential games with potential functions shown in Figure 5.12.

Γ_1	*A*	*B*
a	(3,3)	(0,5)
b	(5,0)	(1,1)

Γ_2	*A*	*B*
a	(1,0)	(1,1)
b	(0,0)	(0,1)

Figure 5.11 Exact Potential Games

V_1	*A*	*B*
a	0	2
b	2	3

V_2	*A*	*B*
a	1	2
b	0	1

Figure 5.12 Exact Potential Functions

If a third normal form game is formed as $\Gamma_3 = \Gamma_1 + \Gamma_2$, then Γ_3 is an exact potential game with exact potential $V_3 = V_1 + V_2$.

$\Gamma_3 = \Gamma_1 + \Gamma_2$	*A*	*B*
a	(4,3)	(1,6)
b	(5,0)	(1,2)

$V_3 = V_1 + V_2$	*A*	*B*
a	1	4
b	2	4

Example 5.13: Linear Combination of Ordinal Potential Games[5]

Consider the finite normal form games shown in Figure 5.13. These games are ordinal potential games with potential functions shown in Figure 5.14.

Γ_1	*A*	*B*
a	(3,3)	(0,5)
b	(5,0)	(1,1)

Γ_2	*A*	*B*
a	(8,0)	(0,-1)
b	(0,2)	(4,0)

Figure 5.13: Ordinal Potential Games

V_1	*A*	*B*
a	0	2
b	2	3

V_2	*A*	*B*
a	3	0
b	2	1

Figure 5.14: Ordinal Potential Functions

If a third normal form game is formed as $\Gamma_3 = \Gamma_1 + \Gamma_2$, shown in Figure 5.15, then Γ_3 is not an ordinal potential game as it contains an improvement cycle. Thus the space of ordinal potential games is not closed under addition. Further as Γ_1 is an exact potential game, it is

[5] A similar example is shown in [Voorneveld_96].

seen that an additive combination of an exact potential game and ordinal potential game is not guaranteed to be a potential game either. This holds significant meaning for the identification of ordinal potential games as unlike exact potential games we cannot show that a game is an ordinal potential game merely by showing that it can be expressed as a sum of ordinal potential games. Instead, we have to show that the entire game is better response equivalent to some exact potential game.

$\Gamma_3 = \Gamma_1 + \Gamma_2$	A	B
a	(11,3)	(0,4)
b	(5,2)	(5,1)

Figure 5.15: A Game Formed by Additive Combination of Ordinal Potential Games.

Example 5.14: Target SINR Power Control[6] (*)

Consider a single cell network of cognitive radios adjusting their transmit powers in an attempt to achieve a target SINR at a common base station. In general this can be modeled as a myopic repeated game as follows. The set of power adapting cognitive radios form the player set. Each player's action set is defined by the associated radio's available power levels, $P_i = [0, p_{max}]$. Each radio's adaptations are guided by the utility function shown in (5.25)

$$u_i(\mathbf{p}) = -\left|\hat{g} - \frac{g_i p_i}{1/K\left(\sum_{k \in N \setminus i} g_k p_k + s\right)}\right| \tag{5.25}$$

where $p_i \in P_i$, $\mathbf{p} = (p_1, \ldots, p_n)$ is a transmit power vector, g_i is the gain from radio i to the base station, K is the spreading gain, σ is the noise power at the base station, and $\hat{g}$ is the target SINR.

As shown the utilities in this game are not well formed as the first derivative is not defined everywhere and the utility function is not in the form of one of the forms listed in Table 5.5. However, consider the objective function shown in (5.26) which is better response equivalent to (5.25) assuming noise power is greater than zero.

[6] The analysis in this example was originally presented in [Neel_04a].

$$u_i^{'}(\mathbf{p}) = -\left[\hat{g}/K\left(\sum_{k\in N\setminus i} g_k p_k + s\right) - g_i p_i\right]^2 \tag{5.26}$$

Expanding and rearranging (5.26) yields (5.27).

$$\begin{aligned} u_i^{'}(\mathbf{p}) = &-g_i^2 p_i^2 + 2\hat{g}/Ks\, g_i p_i \\ &+ 2\hat{g}/K\left(\sum_{k\in N\setminus i} g_i g_k p_i p_k\right) \\ &-\left[\hat{g}/K\left(\sum_{k\in N\setminus i} g_k p_k + s\right)\right]^2 \end{aligned} \tag{5.27}$$

Now notice that $\left\langle N, P, \left\{-g_i^2 p_i^2 + 2\hat{g}/Ks\, g_i p_i\right\}\right\rangle$ is a self-motivated game; $\left\langle N, P, \left\{2\hat{g}/K\left(\sum_{k\in N\setminus i} g_i g_k p_i p_k\right)\right\}\right\rangle$ is a BSI game with symmetric interaction term $w_{ij}(p_i, p_j) = g_i g_j p_i p_j$ which has been scaled by the factor $2\hat{g}/K$; and $\left\langle N, P, \left\{\left[\hat{g}/K\left(\sum_{k\in N\setminus i} g_k p_k + s\right)\right]^2\right\}_{i\in N}\right\rangle$ is a dummy game. So by Theorem 5.23, the game $\left\langle N, P, \{u_i^{'}\}\right\rangle$ is an exact potential game as it is a linear combination of exact potential games. Because $\left\langle N, P, \{u_i^{'}\}\right\rangle$ is a better response transformation of $\left\langle N, P, \{u_i\}\right\rangle$, by Theorem 5.8, $\left\langle N, P, \{u_i\}\right\rangle$ is an ordinal potential game with a potential function given by (5.28)

$$V(\mathbf{p}) = 2\hat{g}/K\left(\sum_{i\in N}\sum_{i>k}^{n} g_i g_k p_i p_k\right) + \sum_{i\in N}\left(-g_i^2 p_i^2 + 2\hat{g}/Ks\, g_i p_i\right) \tag{5.28}$$

5.4 Steady States of Potential Games

In addition to some valuable convergence and stability properties that will be established in Sections 5.6 and 5.7, potential games have several interesting Nash equilibrium properties.

Theorem 5.24: *Nash equilibrium existence and finite potential games*
All finite potential games have at least one NE.
Proof: As shown in Theorem 5.11 all finite potential games have FIP. As shown in Chapter 4, all games with FIP have at least one NE.

While Theorem 5.24, assures us of the existence of an NE in a finite potential game, it provides little help in identifying the game's NEs. However, Theorem 5.25 provides a powerful result for identifying NE in finite and infinite potential games.

Theorem 5.25: *Potential function maximizers and Nash equilibria*
Given a potential game, $\Gamma = \langle N, A, \{u_i\} \rangle$ with potential function *V*, global maximizers of *V* are Nash equilibria.
Proof: Suppose $a^* = \max_{a \in A} V(a)$ is not a NE. Then there is some $a' \in A$ where *a'* differs from a^* in coordinate *i* such that $u_i(a') > u_i(a^*)$. But this implies that $V(a') > V(a^*)$ and that a^* is not a global maximizer of *V*. Therefore a^* must be a NE.

The global maximizers of the potential function, *V*, may merely be a subset of the all NE in a game. Fortunately, only those NE that are isolated maximizers of *V* are stable for most selfish processes - a claim we prove in Section 5.7.

In Chapter 4 we showed that all games with FIP must have an NE. However, this need not be the case for games with AFIP. Instead games with AFIP are assured of having what is known as an *e-Nash equilibrium* (*e*-NE).

Definition 4.15: *e-Nash equilibrium*
An action tuple, a^*, is said to be an *e*-Nash equilibrium if for every $i \in N$, $u_i(a^*) \geq u_i(a_i, a^*_{-i}) - e \;\forall a_i \in A_i$ where $e \geq 0$.

A game can have an ε-Nash equilibrium without having a Nash equilibrium. For instance, consider Zeno's game introduced in Example 5.8, but with action sets defined as $A_1 = A_2 = [0,1)$. In this scenario, there is no action tuple that cannot be improved upon by increasing a_i meaning that there is no NE in the game However, given ε>0, all action tuples where both components are in the interval $[1-e,1)$ are ε-Nash equilibria. Note that as an NE is an *e*-Nash equilibrium for all ε>0, a game's NE are always a subset of its ε-NE for all for all ε>0.

Theorem 5.26: AFIP and ε-NE Existence
All games with AFIP have at least one ε-NE.
Proof: Given a game Γ with AFIP, there must be at least one action tuple, a^*, from which there exists no unilateral deviation that improves the payoff by at least e (otherwise the game would not have AFIP). This action tuple a^* must be an ε-Nash equilibrium as there exists no other $a \in A$ such that $u_i\left(a_i, a_{-i}^*\right) > u_i\left(a^*\right) + e$.

For generalized ε-potential games, results similar to Theorem 5.24 and Theorem 5.25 can be established for the existence and identification of e-Nash equilibria.

Theorem 5.27: *Generalized e-potential games and e-Nash equilibria* (*)
All generalized ε-ordinal potential games have at least one ε-Nash equilibrium.
Proof: As all generalized ε-potential games have AFIP, all generalized ε-potential games have at least one *e-Nash equilibrium* as AFIP implies the existence of some action tuple action tuple, a^*, for which there exists no $a_i^{'}$ such that $u_i\left(a_i^{'}, a_{-i}^*\right) - e \geq u_i\left(a_i^*, a_{-i}^*\right)\ \forall a_i \in A_i$ (otherwise no ε improvement path would be finite).

Again, maximizers of an *e-potential function* are also *e-NE*.

Theorem 5.28: *Generalized e-potential function maximizers and e-Nash equilibria* (*)
Given a generalized ε- potential game, $\Gamma = \langle N, A, \{u_i\} \rangle$ with potential function *V*, maximizers of *V* are ε-Nash equilibria.
Proof: Suppose $a^* = \max_{a \in A} V(a)$ is not an ε-Nash equilibrium. Then there is some $a' \in A$ where a' differs from a^* in coordinate i such that $u_i(a') > u_i\left(a^*\right) + e_1$. But this implies that $V(a') > V\left(a^*\right) + e_2$ and that a^* is not a maximizer of *V*. Therefore a^* must be an e-Nash equilibrium.

Unfortunately, not all functions can be assured of having a maximum so some conditions must be satisfied to ensure that a maximum of *V* exists.

Theorem 5.29: *Nash equilibria and continuous potential functions* [Monderer_96]
All potential games with a compact action space and a continuous potential function have at least one NE.
Proof: All continuous functions on a compact action space are uniformly continuous and have a global maximum. So if the potential function is continuous and the action space is compact, then the potential function must have a maximum. By Theorem 5.25 this global maximum is an NE.

Note that as all NE are also ε-NE, Theorem 5.29 also supplies a sufficient condition for the existence of ε-NE. Thus for potential games, we have readily applied conditions for determining the existence of and identifying NE and ε-NE for finite and infinite potential games. Beyond directly evaluating the continuity of the potential function, techniques for determining the continuity of a potential game's potential function are considered in Section 5.3.4.

5.5 Optimality

In general, little can be said about the optimality or desirability of the steady states of a cognitive potential game. They need not be Pareto efficient, and they are not generally maximizers of a design objective function. For example if we are trying to maximize the sum utilities of the radios, the steady-state of the cognitive radios' dilemma is undesirable.

However, when the potential function is also the network objective function, i.e., $V = J$, then if V admits a global maximum, then there exists a NE that is optimal. Further, as deterministic unilateral play increases the value of V with each iteration, it is safe to say that the stable steady-state of the network will give better performance than the initial state of the network. For example, consider the adaptive interference avoidance game of Example 5.5. While this game necessarily has numerous steady states, if the network's designer is attempting to minimize total network interference, then the steady-states will generally be desirable and performance will improve with each adaptation.

Alternately, it is a relatively straight forward task for a cognitive radio designer to move the steady-state of an exact potential game, a^* to some other desired valid state, a^{**} by exploiting the linear space properties of exact potential games. The procedure for doing so is as follows.[7]

Introduce an identical interest network cost function, $NC(a)$ which is found by solving (5.29) for $NC(a)$

[7] This procedure is taken from [Neel_02].

$$\frac{\partial V\left(a^{**}\right)}{\partial a_i}+\frac{\partial NC\left(a^{**}\right)}{\partial a_i}=0 \tag{5.29}$$

where $V(a)$ is the original game's potential function. In other words, at the desired action tuple, the network cost function should have the negative slope of the potential function's slope. After solving for $NC(a)$, "charge" the radios this cost function. This can either be done by directly incorporating NC into each radio's reasoning process or as a cost imposed by the network on the radios. Thus the modified utility function for each radio takes the form shown in (5.30).

$$u_i^{'}(a)=u_i(a)+NC(a) \tag{5.30}$$

Since the original game was an exact potential game and $NC(a)$ is an identical interest (coordination) game, the modified game is also an exact potential game with an exact potential given by (5.31).

$$V'(a)=V(a)+NC(a) \tag{5.31}$$

Note that this process may introduce additional NE depending on the characteristics of $V(a)$ and the choice of $NC(a)$. Care should also be taken so that the original NE is no longer a NE. To minimize the creation of new NE, it is suggested that the function be of low order or a piecewise function. Ideally, V should be a concave function and NC a linear function as this combination preserves both the concavity and the uniqueness of the potential maximizing NE. Also note that it is possible to impose arbitrary cost functions to create arbitrary potential functions. However, this will not in general be desirable, as significant alterations in the potential function will result in significant changes to the behavior of the network, perhaps negating the original advantages of the adaptation scheme. Further, while this solution is deterministic, the actual channel conditions will be stochastic and the stability of the NE should also be considered. Thus in general, it is anticipated that small changes in the neighborhood of the original NE will be more desirable than more significant continual alterations to the game. Finally note that while this example considered the addition of an additive cost identical interest function, similar results should be achievable using additive self-motivated functions (such as battery life for a transmit power control algorithm).

5.6 Convergence of Potential Games

Unlike in Chapter 4, potential games permit us to establish convergence results for myopic repeated games for games with both finite and infinite action spaces. Continuing a trend from Chapter 4, by applying traditional engineering analysis tools in a game theoretic setting, we are able to analyze broad classes of cognitive radio algorithms and ontological radio algorithms instead of just a single procedural algorithm at a time.

5.6.1 Decision Rule Classes

As we study the convergence of potential games, it is useful to define some additional decision rules. Continuing the notation from previous chapters, a decision rule is a mapping $d_i : A \to A_i$ that defines the action that a cognitive radio based on an observation of the network state. As before, the convergence analysis of this chapter also constrains itself to individually rational decision rules as defined in Definition 4.3.

In addition to the decision rules considered in Chapter 4, it will be useful to consider two more classes of decision rules - ε-*better response dynamic* and the *intelligently random better response dynamic*.

Definition 5.16: ε-*Better Response Dynamic*
A decision rule $d_i : A \to A_i$ is an *e-better response dynamic* if given ε>0, each adaptation would improve the radio's utility by at least ε if all other radios continued to implement the same wave forms, i.e., $d_i(a) \in \{b_i \in A_i : u_i(b_i, a_{-i}) \geq u_i(a_i, a_{-i}) + e\}$.

In addition to being useful for convergence analysis in infinite potential games, an ε-better response dynamic is also used in Chapters 6 and 7 to stabilize a system which is unstable because of a lack of isolated maximizers of *V*.

Definition 5.17: *Intelligently Random Better Response Dynamic*
A decision rule $d_i : A \to A_i$ is an *intelligently random better response dynamic* if for each $t_i \in T_i$, radio *i* randomly chooses an action from its better response set, i.e., $d_i(a) = rand\{b_i \in A_i : u_i(b_i, a_{-i}) > u_i(a_i, a_{-i})\}$. If the set is empty, then no adaptation occurs.

Intelligently random better response dynamics have certain analytical advantages such as their only steady states are NE and as we will see shortly, these decision rules converge under a broad range of conditions.

5.6.2 Convergence in Finite Games

As we established in Theorem 5.11, having a generalized ordinal potential is equivalent to having FIP for finite games. In Chapter 4, we established the conditions under which a finite game with FIP converges. Adding to these results, it is seen that both the ε-better response dynamic and the intelligently random better response dynamic generate improvement paths which by FIP must be finite. Thus these two decision rules converge under round-robin, random, and asynchronous decision rules. As before, both decision rules, however, can fail under synchronous timing. For instance, consider the game matrix shown in Figure 5.16: Coordination Game. Repetition of a synchronous best response starting at (a,A) yields the oscillation $(a,A)\leftrightarrow(b,B)$ where neither (a,A) nor (b,B) are NE. Finally, it should be noted that even in a finite game, the ε-better response dynamic may not converge to an NE if ε is sufficiently large. For instance, consider again the game of Figure 5.16 with ε=3.

Γ	A	B
a	(0,0)	(3,3)
b	(3,3)	(0,0)

Figure 5.16: Coordination Game With Synchronous Play

These convergence results are summarized in Table 5.6 where an entry of 'Y' specifies convergence and 'N' indicates that convergence is not assured.

Table 5.6: Guaranteed Convergence Conditions for Finite Potential Games

Decision Rules	Timings			
	Round-Robin	Random	Synchronous	Asynchronous
Best Response	Y	Y	N	Y
Exhaustive Better Response	Y	Y	N	Y
Random Better Response[(a)]	Y	Y	Y	Y
Random Better Response[(b)]	Y	Y	N	Y
ε-Better Response[(c)]	Y	Y	N	Y
Intelligently Random Better Response	Y	Y	N	Y

(a) Proposed random better response (b) Random better response of [Friedman_01] (c) Convergence to an ε-NE

In general, finite potential games are guaranteed to converge under every set of timings with rational decision rules except for synchronous timings. However, it is possible for a game to simultaneously satisfy multiple models which can expand its convergence conditions. As these models supply sufficient, but not necessary, conditions for convergence, simultaneously satisfying multiple models implies that all convergence conditions are satisfied. For instance, a game may simultaneously be a potential game and IESDS solvable which would imply that such a game would also be guaranteed to converge under synchronous best responses and the random better response of [Friedman_01]. An example of a game that is both a potential game and IESDS solvable is the cognitive radios' dilemma.

5.6.3 Convergence in Infinite Games

To study the convergence of play in infinite potential games, we make the following assumptions.

- The cognitive radio network can be modeled as a myopic repeated game.
- The stage game is a potential game with convex, compact A, and potential function V.

5.6.3.1 Implications of FIP (*)

As we showed in this chapter and in Chapter 4, games with infinite action spaces can have FIP. As these games have FIP, all sequences of better responses must converge in a finite number of steps. Thus all round-robin and randomly timed exhaustive better and best response decision rules must converge to an NE in infinite potential games.

Similarly, under asynchronous timing with a best response or exhaustive better response, the system must eventually happen upon one of the timing sequence of finite length that converges. Likewise ε-better response dynamic and the intelligently random better response dynamic generate improvement paths which by FIP must be finite. Thus these two decision rules also converge under round-robin, random, and asynchronous decision rules.

Unlike before, the random better responses cannot be guaranteed to converge to an NE using the earlier techniques applied to games with FIP. As in the examples of infinite games with FIP, it may be the case that specific actions are required for improvement. In an infinite action space, where a player is randomly selecting its actions from an infinite set, each player would have a theoretically 0% probability of randomly choosing the single action that improves its performance. So the random better response decision rules could not be guaranteed to converge for any class of decision timings. So at odds with the results for finite games, the random better response decision rules converge to an NE under the fewest timings. However, these algorithms are still guaranteed to converge as will be shown in a Section 5.6.3.3. These convergence results are summarized in Table 5.7.

Table 5.7: Guaranteed Convergence Conditions for Infinite Potential Games with FIP

	Timings			
Decision Rules	Round-Robin	Random	Synchronous	Asynchronous
Best Response	Y	Y	-	Y
Exhaustive Better Response	Y	Y	-	Y
Random Better Response[(a)]	-	-	-	-
Random Better Response[(b)]	-	-	-	-
ε-Better Response[(c)]	Y	Y	-	Y
Intelligently Random Better Response	Y	Y	-	Y

(a) Proposed random better response (b) Random better response of [Friedman_01] (c) Convergence to an ε-NE

5.6.3.2 Implications of AFIP

The fact that all generalized ε-potential games have AFIP implies that all ε-improvement paths are finite in a game with AFIP. Thus the ε-better response converges for round-robin and random timings. Further, as games with AFIP are characterized by some

$L \in \mathbb{N}$ such that no ε-improvement paths is longer than L, the ε-better response also converges to an ε-NE for asynchronous timings as $t \to \infty$. As all games with FIP also have AFIP, all decision rules which are not guaranteed to converge under FIP are also not guaranteed to converge under AFIP. Specifically, the two random better response decision rules are not guaranteed to converge under AFIP.[8]

5.6.3.3 General Convergence[9]

Broader convergence implications can be drawn by introducing Zangwill's Convergence Theorem A which for clarity requires a clarification on when a decision rule is closed.

Definition 5.18: *Closed Set-Valued Function*

A set valued function $f: X \to Y$ is said to be *closed* at $x^* \in X$ if the following pair of conditions

1) $x^* = \lim_{k\to\infty} x_k$, $x_k, x^* \in X$
2) $y^* = \lim_{k\to\infty} y_k$, $y_k, y^* \in Y$

imply $y^* \in f\left(x^*\right)$. The function f is said to be *closed* on X if f is closed for all $x \in X$.

Applying this definition to functions of the form $d: A \to A$, d is a closed set-valued function if and only if $a^* \in \lim_{k\to\infty} a^{k+1} = d\left(a^k\right)$ implies $a^* \in d\left(a^*\right)$. Thus d is closed if and only if all limit points of a recursive application of d are also fixed points of d.

Theorem 5.30: *Zangwill's Convergence Theorem A* [Zangwill_69]

Let d:$A \to A$ determine an algorithm that given a point a^0 generates a sequence $\left(a^k\right)_0^\infty$ through the iteration $a^{k+1} \in d(a^k)$. Let a solution set, $S^* \subset A$, be given. Suppose

8 I am not drawing more general implications in this section because of the following scenario. Suppose that a round-robin sequence of best responses generates an action sequence defined by the harmonic series 1, 1 + 1/2, 3/2 + 1/3, … Such a game would have AFIP as for any ε>0 there would come a point from which no single best response and thus no better response could improve by at least ε>0 (specifically the k^{th} term where $1/k=\varepsilon$). Thus an ε-better response would converge. However, all of the other decision rules would diverge. Unfortunately, I do not know of a normal form game that yields such a sequence of best responses, so this implication is left as a footnote to the discussion of AFIP.

9 The application of Zangwill's Convergence Theorem to potential games was independently developed by the author. However, a later literature survey revealed that [Ermoliev_02] had previously made the application outside of the game theory or wireless literature. However, the broader convergence implications of Zangwill's to the decision rules in D^V (defined in Definition 5.19) represent a novel contribution. Also the averaged best response decision rule, though a well known process, is shown to converge by leveraging Zangwill's in [Ermoliev_02]. However, [Ermoliev_02] requires that u_i be concave when only quasi-concavity is required. This distinction is important for many power control algorithms for which u_i is quasi-concave but not necessarily concave.

(1) All points $\left(a^k\right)_0^\infty$ are in a compact set $S \subset A$.

(2) There is a *continuous function* $\alpha : A \to \mathbb{R}$ such that:

(a) if $a \notin S^*$, then $\alpha(a') > \alpha(a) \forall a' \in d(a)$

(b) if $a \in S^*$, then $\alpha(a') \geq \alpha(a) \forall a' \in d(a)$

(3) d is closed at a if $a \notin S^*$.

Then either the recursion $a^{k+1} \in d(a^k)$ arrives at a solution (fixed point), or the limit of any convergent subsequence of $\left(a^k\right)_0^\infty$ is in S^*.

Proof: A proof of this theorem is given in [Zangwill_69].

It should be noted that while it is based on a monotonically increasing function, Theorem 5.30 says nothing about the optimality or desirability of the solution set S^* and the choice of solution sets may be a function of the starting point, a^0. While many functions which are not closed will also converge, the limit points of the recursive application of these functions are not guaranteed to be fixed points.

For instance, consider the decision rule given by (5.32) on the set A=[0,3].

$$d(a) = \begin{cases} 2 - 0.5|2 - a| & a < 2 \\ 3 - 0.5|3 - a| & a \geq 2 \end{cases} \tag{5.32}$$

Clearly, condition (1) is satisfied as A is compact. (2) is satisfied by $\alpha(a) = a$. However (3) is not satisfied as for all $a < 2$, $\lim_{k\to\infty} a^k \to 2$, yet $d(2) = 2.5$. Thus while d converges for a<2, it does not converge to a fixed point of d.

Before casting this theorem in a manner that is appropriate for potential games, let us first introduce the *strict improvement algorithm set*, D^V, a very broad set of decision update algorithms whose adaptations are positively correlated with the potential function.

Definition 5.19: D^V - *Strict Improvement Set* (*)

Given a repeated game with stage game Γ with potential V, the *strict improvement set* for Γ, D^V is given by the set of all decision rules, d, such that if $a^k \neq a^{k+1}$, then $a^{k+1} \in d\left(a^k\right) \Rightarrow V\left(a^{k+1}\right) > V\left(a^k\right)$.

In effect, a decision update algorithm, d, is a member of the strict improvement set for potential game, Γ, if the sequence of actions formed by the recursive evaluation of d

strictly increases the value of V for all points in the action space other than fixed points of d. By definition all individually rational unilateral applied with unilateral timing (round-robin or random) satisfy the conditions to be a member of the strict improvement set for a potential game's potential function (assuming we eliminate repeated action tuples when radios do not adapt). So the best response, exhaustive better response, random better responses, ε-better response, and the intelligently random better response decision rules would all be members of every potential game's strict improvement set.

Theorem 5.31: *Global Convergence of D^V for an Infinite Action Space* (*)
Given a myopic repeated game with a stage game, Γ, which is a potential game with a convex compact action space A and potential V, if the following conditions hold, then the recursion the recursion $a^{k+1} = d\left(a^k\right)$ converges.

1) V is continuous
2) $d \in D^V$
3) d is closed.

Proof: This is just an application of Theorem 5.30 with $\alpha = V$. Thus d converges to some $a^* \in D^*$ where D^* are the set of fixed points for d.

While the members of D^V are guaranteed to converge to fixed points, there exist members of D^V which have non-NE fixed points. For instance, consider the following decision rule frequently which could be used as part of a local search algorithm in a cognitive radio.

Definition 5.20: *Directional Better Response*
A decision rule $d_i : A \rightarrow A_i$ is a *directional better response dynamic* if at each adaptation, radio i adapts its action according to the following where $K_i \in \mathbb{R}$.

$$d_i\left(a^k\right) = \begin{cases} a_i^k + K_i \dfrac{\partial u_i\left(a^k\right)}{\partial a_i} & u_i\left(a_i^k + K_i \dfrac{\partial u_i\left(a^k\right)}{\partial a_i}, a_{-i}^k\right) > u_i\left(a_i^k, a_{-i}^k\right) \\ a_i^k & otherwise \end{cases}$$

This decision rule satisfies all of the conditions of Theorem 5.31, so play converges to a fixed point of d. However, local maximizers and action tuples close to the top of sufficiently peaked potential functions will also be fixed points of this decision rule. Thus an additional condition is needed to ensure that play converges to an NE, and the obvious condition is that the only fixed points of d are NE.

Theorem 5.32: *Convergence of d^V to an NE* (*)

Given a myopic repeated game with a stage game, Γ, which is a potential game with a compact action space A and potential *V*, if the following conditions hold, then the recursion the recursion $a^{k+1} = d\left(a^k\right)$ converges to an NE.

1) *V* is continuous

2) $d \in D^V$

3) *d* is closed.

4) $a^* = d\left(a^*\right)$ if and only if $a^* = \hat{B}\left(a^*\right)$.

Proof: This is just a variation of Theorem 5.32 where the fixed points of *d* are constrained to the game's NE.

With this fourth condition in place, we are assured that the network would actually converge to the steady-state identified from our NE analysis. Examining the six originally considered decision rules (best response, exhaustive better response, random better responses, ε-better response, intelligent better response) only the ε-better response has fixed point other than NE. However, all fixed points of that decision rule are ε-NE, a generally acceptable result.

It is also possible to identify decision rules which, under specific conditions, are members of D^V and whose only fixed points are NE. For example, consider the following decision rule.

Definition 5.21: *Averaged Best Response*

A decision rule $d_i : A \rightarrow A_i$ is an *averaged best response* if at each adaptation, radio *i* adapts its action according to $d_i\left(a^k\right) = l\, a_i^k + (1 - l)\hat{B}_i\left(a_i^k\right)$ where $l \in (0,1)$.

In general, Definition 5.20 cannot be guaranteed to be a member of D^V as it may not improve a player's utility and thus would not increase the potential function. However, if u_i is quasi-concave in a_i for all $i \in N$ then all of its upper level sets are convex and the averaged best response must increase the adapting player's utility. Further, regardless of the quasi-concavity of u_i, the only fixed points of Definition 5.22 are those points where $a_i^k = \hat{B}_i\left(a_i^k\right)$, in other words, an NE. Thus for an infinite potential game, if every player's utility function is quasi-concave then when all players play Definition 5.20, play will converge to an NE.

Thus by applying a well-known theorem from nonlinear programming, we have established conditions under which a broad range of decision rules will converge to an NE when the stage game is a potential game. Interestingly, the conditions under which the decision rules in D^V converge to an NE in infinite games are also sufficient for convergence in a finite game – recall that the only requirement on A is that it is compact and finite sets are compact. Thus we could have shown that the same algorithms converged for finite games under round-robin and random decision timings. However, we would've been unable to show that play converges under asynchronous timings as the algorithms in D^V need not strictly increase V under asynchronous timing – a problem for infinite A as well.

Example 5.15: Convergence of a Power Control Potential Game

In Example 5.14, we considered a single cell network of cognitive radios adjusting their transmit powers in an attempt to achieve a target SINR at a common base station where each radio, i, can choose any power level from the set $P_i = [0, p_{max}]$ as guided by the utility function shown in (5.33)

$$u_i(\mathbf{p}) = -\left| \hat{g} - \frac{g_i p_i}{1/K \left(\sum_{k \in N \setminus i} g_k p_k + s \right)} \right| \tag{5.33}$$

where $p_i \in P_i$, $\mathbf{p} = (p_1,\ldots,p_n)$ is a transmit power vector, g_i is the gain from radio i to the base station, K is the spreading gain, σ is the noise power at the base station, and $\hat{g}$ is the target SINR. It was shown earlier that this game is an ordinal potential game with a potential function given by (5.34).

$$V(\mathbf{p}) = 2\hat{g}/K \left(\sum_{i \in N} \sum_{i > k}^{n} g_i g_k p_i p_k \right) + \sum_{i \in N} \left(-g_i^2 p_i^2 + 2\hat{g}/Ks\, g_i p_i \right) \tag{5.34}$$

Thus it is expected that the following decision rules converge: best response, exhaustive better response, random better responses, intelligently random better response, and ε-better response all converge. Further as (5.33) is quasi-concave in p_i, the averaged best response is also predicted to converge.

A simulation of this network with seven mobiles targeting a linear SINR of 2.71 (which corresponds to a BER of about 10^{-2} for BPSK signals) operating at 5 GHz, randomly distributed about a 4 km^2 with a spreading gain of 100, a path loss exponent of 3 confirms these claims. Based on the mobile distribution shown in Figure 5.17, Figures 5.15 through Figure 5.22 depict the behavior of the network with round-robin timing and best-response, averaged best-response (λ=0.5), intelligently random better response, random better response, and ε-better response, respectively. In these five figures, the top plot shows the value of the objective function and the lower plot depicts the change in power levels over time. Note that all five decision rules converge to the same equilibrium[10] with the best response algorithm converging the fastest and the random better response the slowest – supporting the supposition of Chapter 4 that when convergent deterministic algorithms tend to converge at faster rates than random algorithms.

[10] Technically, the ε-better response algorithm is converging to a point near the NE, but at this resolution it is difficult to see the difference.

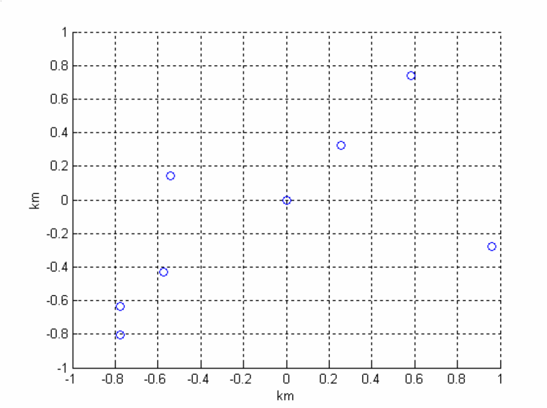

Figure 5.17: Radio Distribution

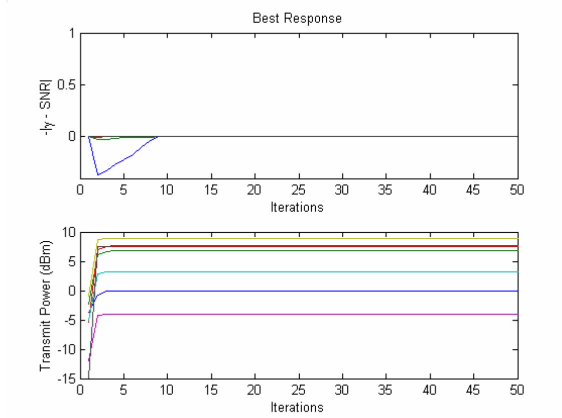

Figure 5.18: Network Behavior Under Best Response Decision Rules

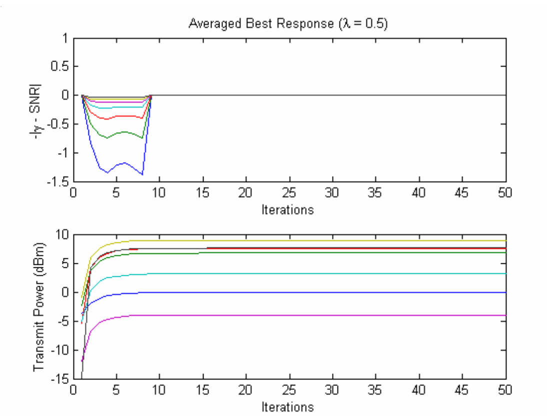

Figure 5.19: Network Behavior Under Averaged Best Response Decision Rules

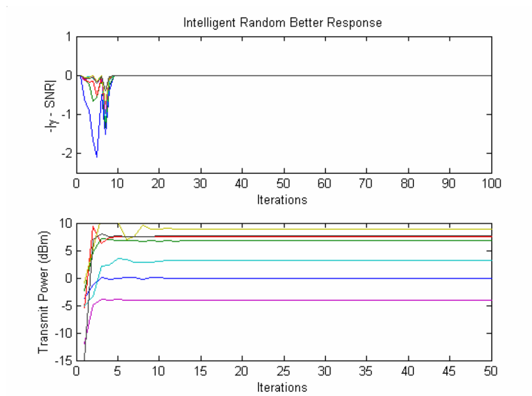

Figure 5.20: Network Behavior Under Intelligent Random Better Response Decision Rules

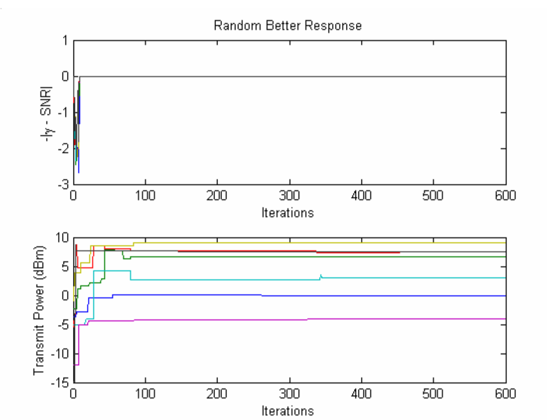

Figure 5.21: Network Behavior Under Random Better Response Decision Rules

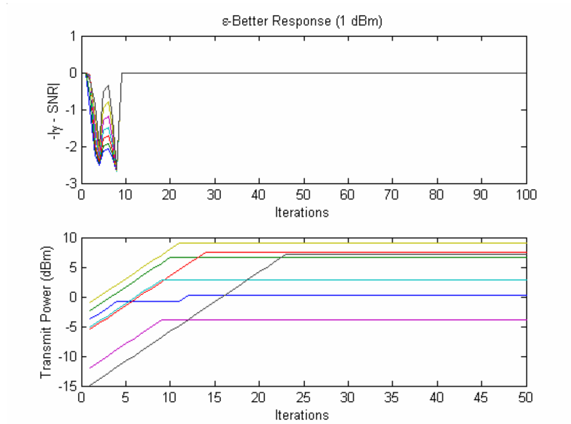

Figure 5.22: Network Behavior Under ε-Better Response Decision Rules

5.6.4 Convergence Rate (*)

Both FIP and AFIP are characterized by integers, L_{FIP} and L_{AFIP}, that bound the length of the longest improvement and ε-improvement path. L_{FIP} can be quickly bounded by |A| and L_{AFIP} by $\left(V_{\max} - V_{\min}\right)/e_V$ for a generalized ε-potential game with a potential function bounded by V_{min} and $V_{\max}$ and minimum potential function increase e_V. For round-robin decision timings, no more than $|N|L_{FIP}$ or $|N|L_{AFIP}$ iterations can be made for any deterministic better response or ε-better response decision rule, respectively. These particular bounds on the number of iterations comes from an assumption of an initial action vector such that $V\left(a^0\right) = V_{\min}$ and the requirement that for every $|N|$ iterations at least one player must be able to improve its payoff for games with FIP and improve by at least an e in games with AFIP as otherwise an NE or an ε-NE has been reached, respectively.

5.7 Impact of Noise and Stability

As covered in Chapter 4, any practical implementation of a cognitive radio network occurs in a noisy environment which corrupts the observation process. Given cognitive radio network $\Gamma = \left\langle N, A, \{u_i\}, \{d_i\}, T \right\rangle$, the noise corrupted cognitive radio network is modeled by $\Gamma = \left\langle N, A, \{\tilde{u}_i\}, \{d_i\}, T \right\rangle$ where $\tilde{u}_i$ is defined as shown in (5.35)

(*Noisy utility*) $$\tilde{u}_i\left(a,t\right) = u_i\left(a\right) + n_i\left(a,t\right) \tag{5.35}$$

where $n_i(a,t)$ is a stochastic process corrupting the evaluation of the player's utility function at time t. So when a rational decision maker believes that $\tilde{u}_i\left(b_i, a_{-i}, t\right) > \tilde{u}_i\left(a_i, a_{-i}, t\right)$, it may be because b_i is a better choice, i.e., $u_i\left(b_i, a_{-i}\right) > u_i\left(a_i, a_{-i}\right)$, or because noise has corrupted the observation at time t. For our purposes, this means that instead of implementing b_i as would have normally been predicted, the radio may implement a_i.

5.7.1 Operational State Characterization

Under the reasonable assumption that $n_i(t)$ is unbounded (perhaps because the noise source is Gaussian) and u_i is bounded, then there is a nonzero (though perhaps very small) probability that $\tilde{u}_i(b_i, a_{-i}, t)$ is less than $\tilde{u}_i(a_i, a_{-i}, t)$ regardless of how much greater $u_i(b_i, a_{-i})$ is than $u_i(a_i, a_{-i})$. So under normal operating conditions, a cognitive radio always has a nonzero chance of making a mistake and the network has a theoretically nonzero chance of ending up in any state in the network. As covered in Chapter 4, this implies that any typical noisy cognitive radio network can be modeled as an ergodic Markov chain under any myopic individually rational decision rule and any of the considered decision timings except for a round-robin timing which violates the aperiodicity condition.

Thus we could expect a cognitive radio network to wander aimlessly about its states, but as we saw in Chapter 4, cognitive radio networks tend to have a higher probability of occupying an NE than other states. This trend is especially true for NE in potential games that are isolated potential maximizers as they are Lyapunov stable for myopic individually rational decision rules.

5.7.2 Lyapunov Stability of Potential Games

Almost as valuable of a result as its strong convergence properties, potential games are also characterized by strong stability properties. Specifically all decision rules with isolated fixed points that satisfy Theorem 5.32 are Lyapunov stable and thus asymptotically stable.

5.7.2.1 Related Work

To date, the game theory literature has treated stability of potential games as a continuous time phenomenon.[11] For example, [Slade_94] showed that in a Cournot oligopoly there existed a "Fictitious objective function" $F : A \rightarrow \mathbb{R}$ that increased with every unilateral deviation, which as [Monderer_96] points out, makes the game a potential game. When considering stability [Slade_94] considered a continuous time directional better response dynamic given by (5.36).

$$\dot{a}_i = da_i / dt = \partial u_i(a) / \partial a_i = h_i(a) \qquad (5.36)$$

[11] Other authors, e.g., [Hofbauer_01] and [Sandhom_01], have studied stability in the context of continuous player sets – a reasonable approximation for evolutionary dynamics of biological systems, but not generally a reasonable assumption for cognitive radio networks.

[Slade_94] then showed that since $J(a)=\left[\partial h_i(a)/\partial a_j\right]=H(a)=\left[\partial F(a)/\partial a_i \partial a_j\right]$, H is symmetric, and that H's determinant is nonzero, its characteristic roots are all real and nonzero which then implies that all maximizers of F are locally asymptotically stable.

[Anderson_99] generalizes the results of [Slade_94] to all infinite potential studies with compact action spaces under round robin directional better responses showing that the process is Lyapunov stable for continuous time adaptations. However, [Anderson_99] extends this result by introducing the Fokker-Planck given in (5.37)

$$\frac{\partial F_i(a,t)}{\partial t}=-E\left[\frac{\partial u_i}{\partial a_i(t)}(a,t)\right]f_i(a,t)+\frac{s^2}{2}f_i^{'}(a,t)\ \forall i\in N \tag{5.37}$$

where $E[\cdot]$ is the expectation operator, $F_i(a,t)$ is the probability that player i chooses an action less than or equal to a_i, and $f_i(a,t)$ is the pdf that corresponds to $F_i(a,t)$. While (5.37) could be used to find a steady state distribution by evaluating (5.37) as $t\to\infty$, this can be quite tedious and then is only applicable to continuous time directional improvement algorithms corrupted by Gaussian noise processes.

A more promising result is given in [Hicks_04b] which examines a noisy repeated potential game operating with a round robin best response decision update algorithm and shows that the algorithm almost surely converges to a region around a Nash equilibrium under the following conditions:

1) Adaptations are corrupted by a noise process bounded by $d>0$
2) V has no local maxima
3) All global maxima of V are isolated
4) V is strictly increasing for all $a\in A$ such that $\left\|a^*-a\right\|<h$ where $h>0$.

In effect, [Hicks_04] indirectly exploits the Lyapunov stability of a round robin best response action tuple update algorithm to ensure that if play is disturbed by no more than an $d>0$, then play converges to an $e>0$ of a potential maximizing NE.

5.7.2.2 Stability of Potential Games (*)

Unfortunately, the decisions made by people and the decisions made by cognitive radios are not made in continuous time. Decisions such as "I'll go to the burger joint instead of the sub shack today," or "I'll implement FM instead of BPSK," are discrete events. However, continual adaptations in continuous time make sense in control systems and analog circuitry which is why the continuous time version of Lyapunov stability makes sense in a controls setting. For cognitive radios implemented in digital circuits, a discrete time version of Lyapunov's theorem is needed which we presented in Chapter 3 and reproduce in the following.

Theorem 5.33: *Lyapunov's Direct Method for Discrete Time Systems* ([Medio_01] Theorem 3.4)

Given a recursion $a\left(t^{k+1}\right)=d^t\left(a\left(t^k\right)\right)$ *with fixed point* a^*, *we know that* a^* *is Lyapunov stable if there exists a continuous function (known as a* Lyapunov function*) that maps a neighborhood of* a^* *to the real numbers, i.e.,* $L:N(a^*)\rightarrow\mathbb{R}$, *such that the following three conditions are satisfied:*

1) $L\left(a^*\right)=0$
2) $L(a)>0\ \forall a\in N\left(a^*\right)\setminus a^*$
3) $\Delta L\left(a(t)\right)\equiv L\left[d^t\left(a(t)\right)\right]-L\left(a(t)\right)\le 0\ \ \forall a\in N\left(a^*\right)\setminus a^*$

Further, if conditions 1-3 hold and

a) $N\left(a^*\right)=A$, *then* a^* *is globally Lyapunov stable;*
b) $\Delta L\left(a(t)\right)<0\ \forall a\in N\left(a^*\right)\setminus a^*$, *then* a^* *is asymptotically stable;*
c) $N\left(a^*\right)=A$ *and* $\Delta L\left(a(t)\right)<0\ \forall a\in N\left(a^*\right)\setminus a^*$, *then* a^* *is globally asymptotically stable.*

Now consider the function given by (5.38).

$$L^V(a)=-V(a)+V\left(a^*\right) \tag{5.38}$$

where V is a bounded continuous potential function with isolated potential maximizer a^*. In the neighborhood of a^*, (5.38) satisfies conditions 1 and 2 for being a Lyapunov function. Now consider any decision rule in D^V played on a potential game with potential function V. As this is a potential game, $V\left(a^{t_{k+1}}\right) \geq V\left(a^{t_k}\right)$ so $V\left[d^{t_k}\left(a^{t_k}\right)\right] - V\left(a^{t_k}\right) \geq 0$ and $\Delta L\left(a^{t_k}\right) \leq 0 \ \forall a \in N\left(a^*\right) \setminus a^*$ satisfying condition 3.

Thus L^V is a Lyapunov function for any member of D^V with round robin or random timing that can be modeled as a potential game with a bounded continuous potential function. Further, all maximizers of V (our method for finding NE in a potential game) are also Lyapunov stable. Also note that if the game has a unique NE, then a^* is globally Lyapunov stable. And if the decision rule fits into one of the classes of algorithms that deterministically converge discussed in Section 15.4.1.3, then the algorithm is asymptotically stable as well.

Theorem 5.34: *Stability of isolated maximizers of V under D^V (*)*
Given potential game $\Gamma = \left\langle N, A, \{u_i\} \right\rangle$ with potential V, if a^* is an isolated global maximizer of V, then a^* is asymptotically stable for all decision rules in D^V.
Proof: A sufficient Lyapunov function is given by (5.38).

With this result we know that not only are specific decision rules Lyapunov stable (under continuous adjustments), but an entire class of decision rules is Lyapunov stable. So by examining the goals and actions of the radios and applying the potential game modeling criteria presented in this chapter, we can immediately know that a broad class of decision rules are convergent and stable and have readily identified steady-states. We illustrate this result in the following example.

Example 5.16: Stability of a Power Control Potential Game

Consider again the network of Example 5.15 which we previously showed to be a potential game and convergent under the following decision rules: best response, averaged best response, exhaustive better response, random better responses, intelligently random better response, and ε-better response all converge.

Now suppose that noise is present in the system so the radios are no longer observing their true SNR and are instead observing their SNR corrupted by additive Gaussian noise. Keeping the same parameters as before including the mobile distribution shown in Figure 5.17, Figures 5.15 through Figure 5.22 depict the noise-corrupted behavior of the network with round-robin timing and best-response, averaged best-response (λ=0.5), intelligently random better response, random better response, and ε-better response, respectively.

Note that while there exists a theoretical probability of the network operating in any state, the globally and asymptotically stable steady-state and the region closest to it is so much more probable that the network never wanders far from the steady-state implying that isolated potential maximizers are especially valuable for predicting the behavior of a cognitive radio network.

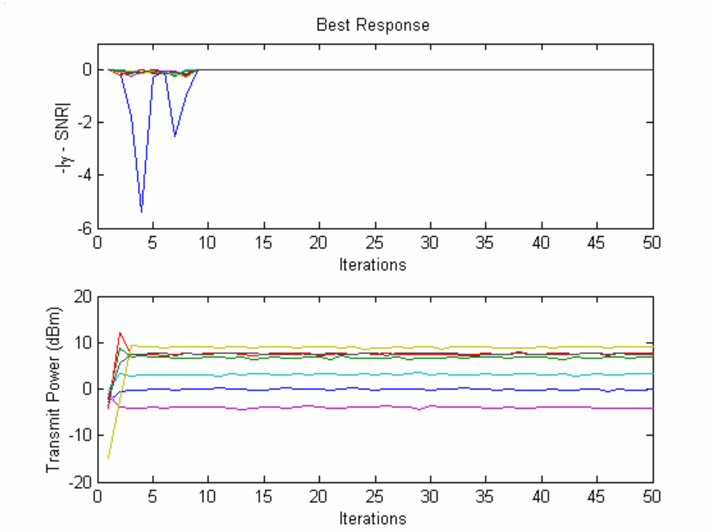

Figure 5.23: Network Behavior Under Best Response Decision Rules

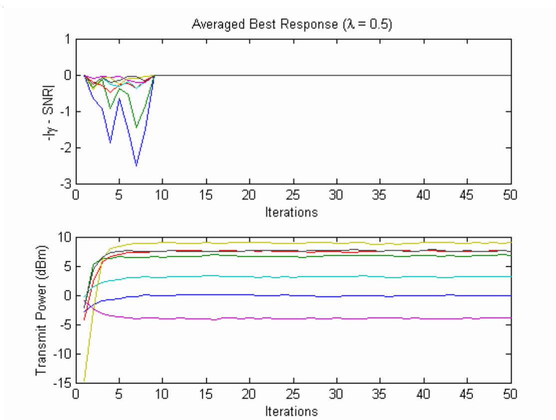

Figure 5.24: Network Behavior Under Averaged Best Response Decision Rules

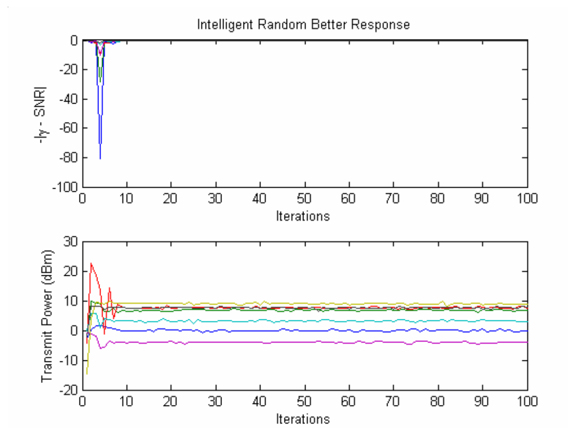

Figure 5.25: Network Behavior Under Intelligent Random Better Response Decision Rules

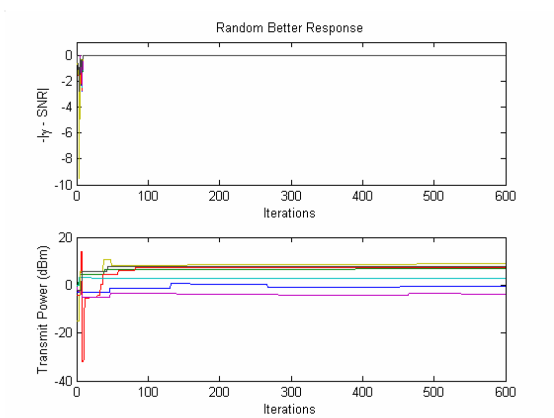

Figure 5.26: Network Behavior Under Random Better Response Decision Rules

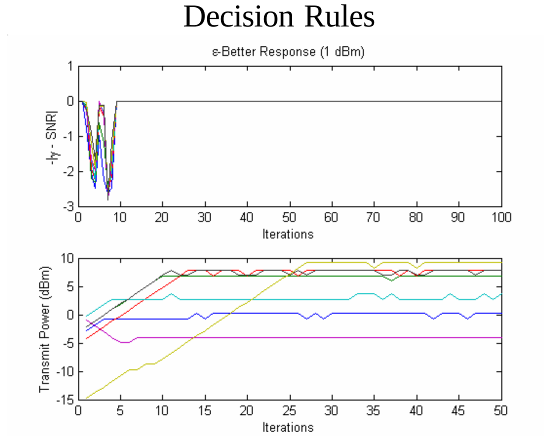

Figure 5.27: Network Behavior Under ε-Better Response Decision Rules

5.8 Analysis Summary and Design Implications

This chapter established several powerful insights for analyzing the interactions of cognitive radios when the radios' goals and actions satisfy the conditions of a potential game. These analysis insights are summarized in Section 5.8.1 and then applied as design guidelines for cognitive radio in Section 5.8.2.

5.8.1 Analysis Summary

In Chapter 4, we identified FIP as an important property for cognitive radio networks as it provides broad conditions for convergence and assures us of the existence of an NE. However in Chapter 4, identifying that a game has FIP required either an exhaustive listing of improvement paths or an exhaustive search to confirm that the stage game lacked improvement cycles. This chapter introduced potential games which are coincident with the set of games that have FIP and techniques for identifying potential games. Thus the techniques presented in this chapters allow us to efficiently apply the implications of FIP to cognitive radio analysis.

Beyond the implications of FIP, each potential game is characterized by a function (the potential function) that monotonically increases with every unilateral selfish adaptation. When coupled with a compact action space and the continuity of its monotonic function, a potential game is assured of convergence and stability.

5.8.1.1 Steady States

As noted in the preceding, finite potential games have FIP which implies the existence of an NE. But infinite potential games with continuous potential functions on a compact action space are also assured of the existence of an NE. Whether a finite or an infinite potential game, compactness of A and continuity of the potential function enable the analyst to identify NE by solving for the maximizers of V (Theorem 5.25). This chapter also considered another equilibrium concept, the ε-NE. It was shown that every NE is also a ε-NE and that ε-NE can again be identified from the maximizers of V (Theorem 5.26).

5.8.1.2 Optimality

In general, little can be said about the optimality or desirability of the steady states of a potential game. They need not be Pareto efficient, and they are not generally maximizers of a design objective function. For example if we are trying to maximize the sum utilities of the radios, the steady-state of the cognitive radios' dilemma is undesirable. Thus, potential games do not possess any special properties that ensure that steady-state behavior will be desirable.

5.8.1.3 Convergence

It was shown that all finite potential games have FIP. Thus based on the results from Chapter 4, it is known that play converges for all myopic self-interested algorithms under round-robin, random, and asynchronous timings. Further, the random decision rule proposed in Chapter 4 also converges under synchronous timing.

For potential games with infinite action spaces, it was shown that some infinite games have FIP which implied the convergence of all myopic self-interested, deterministic, and exhaustive decision rules. However, except for the intelligently random decision rule, the random decision rules could not be guaranteed to converge based on the FIP property. For potential games with AFIP, it was seen that all round-robin, random, and asynchronous ε-better responses converge. Finally, for infinite action spaces, it was shown that all closed decision rules which were members of D^V converge and specifically to converge to an NE if their only fixed points are NE.

The convergence results presented in this Chapter are summarized in Table 5.8 where each entry lists the conditions that ensure the convergence of the associated decision rule class and decision timing. Note that the widest set of convergence conditions hold when round-robin or random timing for which a very broad number of classes of decision algorithms converge while convergence under synchronous timing is only guaranteed for games with FIP implementing the random better response decision rule proposed in Chapter 4.

Table 5.8: Convergence Criteria for Potential Games

Decision Rules	Timings: Round-Robin	Random	Synchronous	Asynchronous
Best Response	1,2,4	1,2,4	-	1,2
Exhaustive Better Response	1,2	1,2	-	1,2
Random Better Response[(a)]	1,2,4	1,2,4	1,2	1,2
Random Better Response[(b)]	1,2	1,2	-	1,2
ε-Better Response[(c)]	1,2,3,4	1,2,3,4	-	1,2,3
Intelligently Random Better Response	1,4	1,4	-	1,2
Directional Better Response[(c)]	4	4	-	-
Averaged Best Response[(d)]	3,4	3,4	-	-

(a) Definition 4.12, (b) Definition 4.13, (c) Convergence to an ε-NE, (d) u_i quasi-concave in a_i
1.Finite game, 2. Infinite game with FIP, 3. Infinite game with AFIP, 4. Infinite game with bounded continuous potential function (implication of D^V)

5.8.1.4 Noise

While Chapter 4 illustrated that the unbounded noise of a wireless network ensures that a cognitive radio network has a nonzero chance of occupying any state, potential games limit these probabilities as all isolated maximizers of the potential function are Lyapunov stable for all decision rules in D^V. Thus the network has a strong tendency to remain in the region about these isolated maximizers even with unbounded noise. So in addition to being steady-states, the maximizers of V are stable under most convergent decision rules.

It was seen that noise can significantly impact the behavior of cognitive radio networks. While game theory typically treated errors in play as being implementation errors – a reasonable assumption for humans – the source of errors for cognitive radios is more likely to be caused by observations being corrupted by noise. Regardless of the source, these errors lead to the Markov models of the networks changing from absorbing Markov chains to ergodic Markov chains. This ultimately has the meaning that for networks that can be modeled as myopic repeated games, the presence of noise means that the network has a theoretically nonzero chance of passing through every possible network state. However, the original absorbing states tend to remain the most commonly visited states in the network. So even with noise in the system, the Nash equilibrium concept (absorbing states for games with weak FIP) retain significant power for predicting the state of the network.

5.8.2 Design Implications

As [Neel_04b] emphasizes, constructing a cognitive radio network so it is a potential game simplifies many design challenges - equilibria are readily identifiable and virtually every self-interested algorithm converges to a stable equilibrium. Because ensuring convergence and stability in a potential game merely requires that the cognitive radios employ self-interested adaptations (to be a member of D^V), very simple decision rules can be employed. Both deterministic (for convergence speed) and random (so a single structure can be used for both FIP and weak FIP games) decision rules can be employed implying that procedural, ontological, loner, and social radios can all be expected to converge to a stable equilibria and are appropriate for use in a potential game cognitive radio network.. Because of this broad range of acceptable implementations and algorithms, potential games permit the implementation of the least complex radios [Neel_04a].

While many different game models were shown to be potential games in this chapter, there is a common theme to many potential games – shared outcomes. Whether shared as an identical interest coordination function or shared via numerous bilateral symmetric or multilateral symmetric interaction terms, there is an implicit sharing of outcomes between pairs, groups, or the entire network of radios. Thus when designing a cognitive radio network to be a potential game, the designer should seek out situations where the radios perceive common outcomes from the interaction of their adaptations. This is known to happen with interference based goals for waveform adaptations (which form BSI games), for network selection algorithms (which are congestion games), target power control SINR algorithms (BSI games) and is exploited in Chapters 6 and 7 to develop powerful cognitive radio networks.

Seemingly at odds with the preceding, the other theme of potential games is a complete lack of interaction, i.e., completely independent outcomes. Functions that model the interaction of transmit power and battery life, error correction coding and BER, and interleaver depth and BER are all examples of goals and adaptations where each adaptation only influences each radio's own goal and would be potential games.

Combining these disparate goals – shared and independent outcomes – into a single goal can help a cognitive radio designer to overcome the primary limitation of potential game cognitive radio networks – it is not assured that a potential game's steady-states will be desirable. As described in [Neel_02b], it is possible to adjust the equilibria of an exact potential game with an undesirable equilibrium to a desirable equilibrium by adding the appropriate self-interested function to the original utility function. For instance, a power control algorithm that transmits too much power can be modified to consider battery life which can lead to less transmitted power and a more desirable outcome. However, unless the original game is an exact potential game this technique may destroy the convergence and stability properties that made the original network attractive. As more potential game cognitive radio networks are ordinal potential games than exact potential games[12] this is problematic for designing potential game cognitive radio networks.

A better approach is to design potential game cognitive radio networks so the potential function is the network design function. When this happens, every unilateral adaptation in the network improves the network performance. Trivially, this can be accomplished by making each radio's goal the network design function. However, as the network design function will typically involve the performance of all radios in the network, this approach necessitates significant message passing and generally will not scale well in a practical implementation. Nonetheless, it is possible to design scalable potential game cognitive radio networks where this occurs without distributing measurements between radios and only from the radios' reacting to their own observations. Such an approach is followed in Chapters 6 and 7 and when possible should be used in the design of cognitive radio networks as it yields the best possible result – convergent, stable algorithms with desirable equilibria that perform for a broad range of implementations.

[12] This is tautologically true as all exact potential games are also ordinal potential games. However, many existing applications of potential games to cognitive radios are ordinal potential games and not exact potential games, so this statement does hold some significance.

5.8.3 Applications

Because of their powerful implications, numerous physical layer potential game applications have been developed since potential games were first proposed for the design of wireless and cognitive radio networks in [Neel_02a]. The following summarizes some of the wireless algorithms that been proposed based on potential game theory.

5.8.3.1 Power Control Applications

[Neel_02b] considers games termed *separable SINR games* wherein each radio can separate its utility function into a function of received signal strength less a function of interference such that the radio's objective function takes the form shown in (5.39)

$$u_i(\mathbf{p},w) = f_{i,1}\left(p_i, g_{i,n_i}, w_i\right) - f_{i,2}\left(\sum_{j\in N\setminus\{i,n_i\}} p_j g_{j,n_i} r_{ij} + s_{n_i}\right) - c_i\left(p_i, w_i\right) \qquad (5.39)$$

where p_i is the transmit power of radio i, w_i is the waveform used by radio i, s_{n_i} is the noise power at the *node of interest* of radio i, v_i (where radio i is measuring (5.39)) and r_{ij} is the fraction of the received signal power transmitted by radio j that interferes with the signal of radio i. This is a widely applicable model to cognitive radios in ad-hoc networks as many adaptive modulation schemes perform their adaptation based on SINR estimates, and can be readily put in this form by working with SINR estimates in dB or by directly subtracting interference from noise. Also note that the nature of the functions of Received Signal Strength (RSS) and interference can be defined in arbitrary ways.

For pure power control games [Neel_02b] limited each radio to only adjusting its power level in response to changes in SINR at its node of interest. In this model, the waveform selected at link initialization remains fixed. Every radio's performance is impacted by interference and chooses an action to change its power level in response to SINR and maintains some minimum threshold, e_i. Additionally, each radio has some cost function associated with each power level, e_i. In this case each radio's objective function can be written as shown.

$$u_i(a) = f_{i,1}\left(p_i, g_{i,n_i}\right) - f_{i,2}\left(\sum_{j\in N\setminus\{i,n_i\}} p_j g_{j,n_i} r_{ij} + N_{n_i}\right) - e_i - c_i\left(e_i\right).$$

This game can be verified to have a potential as shown.

$$\frac{\partial^2 u_i}{\partial a_i \partial a_j} = \frac{\partial^2 u_j}{\partial a_i \partial a_j} = 0, \forall i, j \in M, i \neq j$$

By recognizing that $f_{i,2}$ is a dummy function, a potential function can be written as shown.

$$V(\mathbf{p}) = \sum_{i \in N} \left[f_{i,1}\left(p_i, g_{i,n_i}\right) - c_i\left(p_i\right) \right]$$

[Neel_04a] introduced the potential game analysis of target single-cell SINR games presented in Example 5.14 showing that such an algorithm is a potential game. [Neel_04a] continued to show that power control algorithms guided by target throughput goals and throughput maximization goals are also potential games via better response equivalences.

[Fattahi_05] proposes a power control algorithm which can be modeled by the game $\langle N, \mathbf{P}, \{u_i\} \rangle$ where u_i is given by (5.40)

$$u_i(\mathbf{p}) = \log\left(1 + K \frac{g_i p_i}{1 + \sum_{k \in N \setminus i} g_k p_k}\right) - c_i\left(p_i\right) \qquad (5.40)$$

and claims that (5.40) is an exact potential for the game.

$$V(\mathbf{p}) = \log\left(1 + K \sum_{i \in N} g_i p_i\right) - \sum_{i \in N} c_i\left(p_i\right) \qquad (5.41)$$

However, this is not an exact potential for this game as (5.42) ≠ (5.43), an unsurprising result as the game is actually an example of the sum of an ordinal potential game and an exact (self-interested) potential game which we showed in Example 5.13 to not be closed under addition.

$$\begin{aligned} &V\left(p_i^1, p_{-i}\right) - V\left(p_i^2, p_{-i}\right) = c_i\left(p_i^2\right) - c_i\left(p_i^1\right) \\ &+ \log\left(1 + K\left(\sum_{k \in N \setminus i} g_k p_k + g_i p_i^1\right)\right) - \log\left(1 + K\left(\sum_{k \in N \setminus i} g_k p_k + g_i p_i^2\right)\right) \end{aligned} \qquad (5.42)$$

$$u_i\left(p_i^1, p_{-i}\right) - u_i\left(p_i^2, p_{-i}\right) = c_i\left(p_i^2\right) - c_i\left(p_i^1\right) + \log\left(1 + K\frac{g_i p_i^1}{1 + \sum_{k \in N \setminus i} g_k p_k}\right) - \log\left(1 + K\frac{g_i p_i^2}{1 + \sum_{k \in N \setminus i} g_k p_k}\right) \quad (5.43)$$

In fact, it is uncertain if a game with (5.40) as its a potential game at all as it fails the second derivative test ($\frac{\partial^2 u_i(\mathbf{p})}{\partial p_i \partial p_j} = \frac{\partial^2 u_j(\mathbf{p})}{\partial p_i \partial p_j}$) as shown in the following.

$$\frac{\partial^2 u_i(\mathbf{p})}{\partial p_i \partial p_j} = \frac{\left(1 + K\frac{g_i p_i}{1 + \sum_{k \in N \setminus i} g_k p_k}\right)\left(-K\frac{g_i g_k}{\left(1 + \sum_{k \in N \setminus i} g_k p_k\right)^2}\right) - \left(1 + K\frac{g_i}{1 + \sum_{k \in N \setminus i} g_k p_k}\right)\left(-K\frac{g_i g_k p_i}{\left(1 + \sum_{k \in N \setminus i} g_k p_k\right)^2}\right)}{\left(1 + K\frac{g_i p_i}{1 + \sum_{k \in N \setminus i} g_k p_k}\right)^2}$$

$$\frac{\partial^2 u_i(\mathbf{p})}{\partial p_i \partial p_j} = \frac{K g_i g_j \left(p_i - 1\right)}{\left(1 + K\frac{g_i p_i}{1 + \sum_{k \in N \setminus i} g_k p_k}\right)^2 \left(1 + \sum_{k \in N \setminus i} g_k p_k\right)^2}$$

$$\frac{\partial^2 u_j(\mathbf{p})}{\partial p_j \partial p_i} = \frac{K g_i g_j \left(p_j - 1\right)}{\left(1 + K\frac{g_j p_j}{1 + \sum_{k \in N \setminus j} g_k p_k}\right)^2 \left(1 + \sum_{k \in N \setminus j} g_k p_k\right)^2}$$

However, if (5.40) were modified as shown in (5.44), then the game is clearly an exact potential game as distributing the logarithm yields a utility function of the form shown in (5.39).

$$u_i(\mathbf{p}) = \log\left(K\frac{g_i p_i}{1 + \sum_{k \in N \setminus i} g_k p_k}\right) - c_i\left(p_i\right) \quad (5.44)$$

[Scutari_06] leverages potential game theory to propose two single cell CDMA power control algorithms with the intent of establishing convergence and stability. In the first, [Scutari_06] equates the game given by $\langle N, \mathbf{P}, \{\log p_i\}\rangle$ with the single cell standard (target SINR) power control algorithm of [Yates_99]. However, the correct potential function is given for this algorithm ($V(\mathbf{p}) = \sum_{i\in N} \log(p_i)$). This is actually an example of the catastrophic power control game that we used as a precautionary tale in Chapter 1 on the importance of the interactions of cognitive radios.

Without noting this major limitation, [Scutari_06] proceeds to propose a second algorithm which is identical to the power control algorithm shown in [Fattahi_05] and repeats the erroneous claim that it is an exact potential game with the same erroneous potential function.[13]

5.8.3.2 Waveform Adaptation Applications

For waveform adaptations [Neel_02b] assumes that the radios are a part of a power controlled star network such that the energy received at the sole access point is the same for each radio. Thus each radio has the same node of interest and only maintains a single link. Waveform adaptation is employed and each radio may select any waveform from its waveform set W_i. The utility function for each radio now takes the form shown in (5.45)

$$u_i(w) = f_{i,1}(w_i) - f_2\left(\sum_{j\in N\setminus\{i,n_i\}} \int w_i w_j + s_{n_i}\right) - e_i - c_i(w_i) \qquad (5.45)$$

where f_2 is constrained to linear functions. By recognizing that f_2 is the sum of BSI terms, an exact potential function for this game can be written as shown in (5.46).

$$V(w) = \sum_{i\in N} \sum_{j=1, j\neq n_i}^{i-1} f_2\left(\int w_i w_j\right) - \sum_{i\in N}\left[c_i(w_i) - f_{i,1}(w_i)\right] \qquad (5.46)$$

[Menon_04] uses this model (and better response equivalent modifications) to study signature sequence adaptations guided by the utility function shown in (5.47) where r_{ij} is

[13] [Scutari_06] also makes the erroneous claim that most CDMA cellular power control algorithms are potential games. This simply has never been shown and is likely not true. Most CDMA cellular power control algorithms are, however, supermodular games as [Altman_03] has shown.

the power of the signal transmitted by radio i as measured at radio j and s_i is the signature sequence transmitted by radio i.

$$u_i(s,r) = -\sum_{j \neq i} \frac{s_i^H s_j s_j^H s_i r_{ji}}{r_{ii}} - \sum_{j \neq i} \frac{s_i^H s_j s_j^H s_i r_{ij}}{r_{jj}} \tag{5.47}$$

This game has a potential given by (5.48) which is actually the negation of the total sum correlation of the signals. This makes the game an example of the interference reducing networks presented in Chapter 6. The existence of a potential function is used to establish the convergence of better and best response decision rules, an illustration of which is shown in Figure 5.28.

$$V(\mathbf{s},\mathbf{p}) = -\sum_{i \in N} s_i^H \left(\sum_{j \in N \setminus i} \frac{s_j s_j^H p_{ji}}{p_{ii}} \right) s_i \tag{5.48}$$

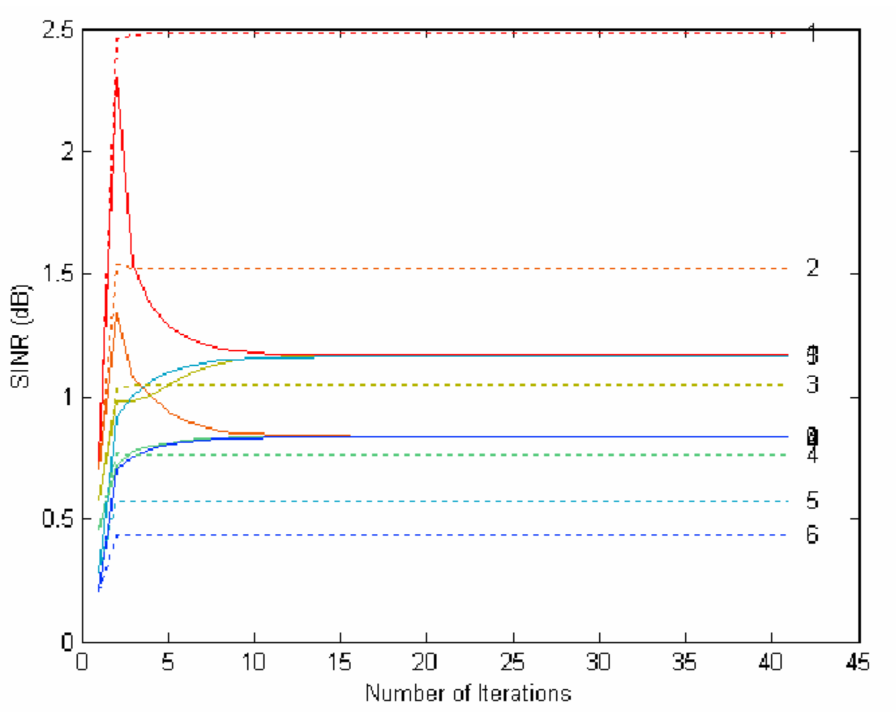

Figure 5.28: SINR levels under Best Response Adaptations of Signature Sequences from Figure 2 in [Menon_04].

[Hicks_04a] uses potential games to design four different signature sequence noisy best response algorithms based on SINR maximization and mean square error (MSE) minimization for correlator and maximum signal to interference and noise ratio (MSINR) receivers. These algorithms are guided by the utility functions shown in Table 5.9 where $\mathbf{R}_{\underline{rr}} \triangleq E[\underline{r}\,\underline{r}^H] = \mathbf{SPS}^H + \mathbf{R}_{\underline{zz}}$. It is shown that the SINR maximization algorithms are ordinal potential games while the MSE Minimization games have weak FIP.[14]

Table 5.9: Utility Functions in [Hicks_04a]

	Correlator	MSINR
SINR Maximization	$u_i(\mathbf{s}) = \frac{p_i}{s_i^T R_{kk}[i] s_i}$	$u_i(\mathbf{s}) = p_i s_i^T R_{kk}^{-1}[i] s_i$
MSE Minimization	$u_i(\mathbf{s}) = -\frac{s_i^H R_{kk}[i] s_i}{s_i^H R_{rr} s_i}$	$u_i(\mathbf{s}) = \frac{-1}{p_i s_i^T R_{kk}^{-1}[i] s_i + 1}$

[Nie_05] considers a frequency selection game with utilities given by (5.49)

$$u_i(f) = -\sum_{j \in N \setminus i} p_j g_{ji} w(f_i, f_j) - \sum_{j \in N \setminus i} p_i g_{ij} w(f_i, f_j) \tag{5.49}$$

where $s(f_i, f_j) = \begin{cases} 1 & f_i = f_j \\ 0 & otherwise \end{cases}$ and shows that it is a potential game and converges under a random best response.

Using potential game theory and the design guidelines of Chapter 6, [Neel_06b] proposes a low complexity dynamic frequency selection algorithm which is shown to converge to a low interference steady-state.

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Chapter 6: Interference Reducing Networks[1]

"…he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention. Nor is it always the worse for society that it was no part of his intention. By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it." – A. Smith, Wealth of Nations

When two people engage in a free trade of resources (perhaps in the form of money), they do so believing that the trade increases their own utility. It is only at best a secondary concern that the trade also increases their trading partner's utility, yet their trading partner's utility must also increase otherwise the partner would not agree to the trade. So barring errors in judgment, it is reasonable to expect that every trade increases both participants' utilities. Beyond just this pair, each trade must also increase the sum of all utilities across society as no one else's resource levels changed. In this manner, while the traders are pursuing their own self-interests they are simultaneously, though unwittingly, pursuing the social interest.

Adam Smith analogized this result to the guidance of an invisible and benevolent hand. Ideally, we want our cognitive radio networks to be guided by their own invisible and benevolent hand. In this chapter we introduce rules for designing potential game cognitive radio networks so the potential function is the network design function – in this case minimizing the interference in the network. As we will see this can be accomplished many different ways including making each radio's goal the network design function. However, as the network design function will typically involve the performance of all radios in the network, this approach necessitates significant message passing and generally will not scale well in a practical implementation. Later in this chapter we introduce a condition we call *bilateral symmetric interference* under which, to take liberties with Adam Smith's analysis, when "each radio intends only its own gain, and it is in this led by an invisible hand to minimize the sum network interference which was no part of its intention."

[1] This chapter is based on [Neel_06a].

When this additional condition is satisfied, we term such cognitive radio networks *interference reducing networks* and we demonstrate that these networks exhibit the following desirable properties:

- Network interference minimizers are steady-states for the system.
- Round-robin, random, and asynchronous self-interested decision rules converge to these steady states.
- Every unilateral self-interested adaptation improves network performance.
- Isolated Lyapunov stability of isolated interference minimizers.

The remainder of this chapter is organized as follows. Section 6.1 presents the basic terminology and modeling parameters used in the remainder of this chapter. Section 6.2 reviews related work in algorithmic and game theoretic papers that will subsequently be shown to satisfy the design rules in later sections. Section 6.3 formally introduces key properties of interference reducing networks. Section 6.4 provides examples of interference reducing networks wherein global knowledge is incorporated into the cognitive radios' goals, including those proposed by other authors. Section 6.5 gives examples of interference reducing networks wherein the radios only incorporate local knowledge. Section 6.6 describes a modified decision algorithm that improves the stability of IRNs. Section 6.7 discusses the performance of interference reducing networks in the presence of legacy devices.

6.1 Modeling and Terminology

The interference reducing network design framework applies to the process of waveform adaptation where a waveform was defined in Chapter 1 to refer to all aspects of a signal utilized in transmitting information between two devices. Thus changing a link's waveform may refer to changes in frequency, modulation, spreading codes, transmission power, error correction, ARQ scheme, etc. One reasonable goal of a waveform adaptation algorithm is to adjust the parameters of a transmitted waveform such that the waveform is predicted to experience a minimal amount of interference.

With this in mind, consider a network of cognitive radios defined by the tuple, $\langle N, \Omega, \{u_i\}, \{d_i\}, T \rangle$ where *N*, $\{u_i\}$, $\{d_i\}$, and *T*, are the same as defined in the cognitive radio model of Chapter 2 and Ω is the action space where the action set of radio *i*, Ω_i, is the collection of waveforms w_i available to radio *i*. Following the notational conventions of the previous chapters, we consider *w* to be a point in the waveform space Ω defined by the vector of waveform choices $(w_1, w_2, \ldots, w_n)$ and write w_{-i} to indicate the waveform - vector of length *n*-1 with component w_i removed, i.e., $w_{-i} = (w_1, w_2, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n)$.

Let $I_i(w)$, $I_i : \Omega \to \mathbb{R}$, represent the interference that cognitive radio *i* observes. From these observations, we can form a measure of the observed global network interference – the *network interference function*, $\Phi(w)$ – by summing these interference terms as shown in (6.1).

$$\Phi(w) = \sum_{i \in N} I_i(w) \tag{6.1}$$

Using (6.1), we can define the key concept of this chapter – the *interference reducing network*.

Definition 6.1: *Interference Reducing Network* (IRN)
A cognitive radio network, $\langle N, \Omega, \{u_i\}, \{d_i\}, T \rangle$, is said to be an *interference reducing network* if for all adaptations the value of Φ decreases, i.e., $d(w) \neq w \Rightarrow \Phi(d(w)) < \Phi(w)$.

6.2 Related Work

Past work on waveform adaptation has concentrated on single-receiver systems (equivalent to the *isolated cluster* scenario considered later in this chapter). A distributed waveform adaptation algorithm for the uplink of a synchronous CDMA system with a single base-station is proposed in [Ulukus_01a] and [Rose_00]. In this algorithm, the system updates the signature sequence, $\boldsymbol{s}_k$, of each user, *k*, in a round-robin fashion where each update is intended to improve the SINR of user *k* at the base-station which is implementing a Minimum Mean Square Error (MMSE) receiver. Specifically, given signature sequence $\mathbf{s}_k(n)$ at iteration n the updated signature sequence is given by (6.2)

$$\mathbf{s}_k(n+1) = \frac{\mathbf{A}_k^{-1}\mathbf{s}_k(n)}{\sqrt{\mathbf{s}_k^T(n)\mathbf{A}_k^{-2}\mathbf{s}_k(n)}} \quad (6.2)$$

where $\mathbf{A}_k = \sum_{j \neq k} \mathbf{s}_j \mathbf{s}_j^T + a^2 \mathbf{I}_n$ where a^2 is the variance of the additive white Gaussian noise at the receiver. It is shown that the round-robin application of (6.2) results in a total monotonically decreasing sequence of total squared correlation (TSC) values where TSC is given by (6.3).

$$TSC = \left(\mathbf{s}_k^T \mathbf{s}_k\right)^2 + 2\mathbf{s}_k^T \left(\sum_{j \neq k} \mathbf{s}_j \mathbf{s}_j^T\right)\mathbf{s}_k + g_k \quad (6.3)$$

where $g_k = \sum_{i \neq k} \sum_{j \neq k} \left(\mathbf{s}_i^T \mathbf{s}_j\right)^2$. As (6.3) is an expression of the interference perceived by each radio in the network, (6.2) is an example of a procedural radio implementation of an IRN. This same algorithm is examined with an asynchronous CDMA systems in [Ulukus_01b], multipath channels in [Concha_01], and multi-carrier systems in [Popescu_02].

Waveform adaptation in networks with multiple collaborative receivers is investigated in [Popescu_04] and [Sung_03b]. Specifically, the waveforms of different mobiles communicating with different base stations are jointly controlled to minimize interference between mobiles. In [Popescu_04], fixed points of greedy waveform adaptation algorithms in these networks are analyzed. In [Sung_03b], the user's utility function is defined in terms of the weighted sum of the interference caused by the particular user at all the receivers in the system; this formulation is then used to prove the existence of NE for the system. These algorithms would be examples of the *globally altruistic IRN scenario* considered later in this chapter.

Technically, this is not the same problem as we have been considering as there is only a single decision maker (the base station) and thus no interactive decision process. However, it is trivial to recast this problem as one where the mobiles are performing this process as other authors have done. For instance, in [Rose_02], presents these same algorithms in a distributed fashion and using a general signal space approach though still with the centralized receiver.

A nonprocedural (and game theoretic) approach is developed in [Hicks_04] (presented in Chapter 5), where waveform adaptation for a centralized network is analyzed from a game theoretic perspective. It is shown that any game where users have one of several combinations of performance metrics (such as Mean Square Error or SINR) and receiver types (such as a correlator or MSINR receiver) results in convergent NE solutions. A similar game theoretic approach is studied in [Menon_04] where multiple users adapt their signature sequences at a common reception point to reduce their own interference. This has the effect of reducing the sum network interference making both [Menon_04] and [Hicks_04] examples of IRNs. [Neel_05] considers a dynamic frequency selection (DFS) algorithm wherein closely located radios are autonomously adapting their frequencies to minimize their perceived interference from one another. Convergence and stability of these algorithms are demonstrated via potential game theory.

We can cast these papers into the operational scenarios identified in this chapter as follows. Specifically, [Hicks_04], [Menon_04], [Ulukus_01a], [Rose_02], [Ulukus_01b], [Popescu_02], [Concha_01] study systems that represent specific instantiations of the *isolated cluster scenario* considered in Section 6.5.1. [Menon_04], [Popescu_04] and [Sung_03a] represent special cases of the *globally altruistic scenario* proposed in Section 6.4.1 [Neel_05] is an example of the *close proximity network scenario* proposed in Section 6.5.2.

Beyond capturing in a single framework many previously proposed waveform adaptation protocols and formalizing operational scenarios that satisfy the proposed policy, this chapter develops new operational scenarios that ensure realization of an Interference Reducing Network, specifically *local altruism* (Section 6.4.2) and *controlled observation processes* (Section 6.5.3). This chapter considers the impact of legacy devices on the proposed policy (Section 6.7). Throughout this chapter, we draw on examples of DFS algorithms and spreading code adaptation to illustrate the operation of these scenarios.

6.3 IRN Properties

While all of the examples of IRNs considered in this chapter are potential games, thereby permitting the characterization of the properties of broad classes of decision rules, a network does not have to be a potential game to be an IRN. For example the MMSE algorithms in [Hicks_04] are IRNs but not potential games[2] and all of the preceding papers, except for [Hicks_04], [Menon_04], and [Neel_05], propose algorithms that have the effect of reducing sum network interference with each adaptation without applying potential game theory. Thus it is useful to briefly define and prove a few key properties of IRNs which are based on the monotonicity of $\Phi(w)$ that results from repeated application of d. Specifically, this section establishes the following properties of IRNs: steady-state existence and optimality, convergence criteria, and stability.

6.3.1 IRN Steady State Properties

As we show in Theorem 6.1, IRNs have attractive steady-state properties.

Theorem 6.1: *Steady-state existence and optimality for IRNs*
Given IRN, $\langle N,\Omega,\{u_i\},\{d_i\},T\rangle$, with continuous $\Phi(w)$ and compact Ω, there exists at least one fixed point for d. Further, at least one steady-state is optimal in the sense of minimizing sum network interference.
Proof: As $\Phi(w)$ is continuous and Ω is compact, there exists some $w^*\in\Omega$ such that $\Phi(w^*)\le\Phi(w)\,\forall w\in\Omega$. Suppose that w^* is not a steady-state, i.e., there is some $\hat{w}\in\Omega\setminus w^*$ such that $\hat{w}=d(w^*)$. Since the network is an IRN, $\Phi(\hat{w})<\Phi(w^*)$ which contradicts the condition placed on w^*. Therefore, w^* must be a steady-state for the network and since it is a minimizer of $\Phi(w)$, w^* is optimal in the sense of minimizing network interference.

Theorem 6.1 also implies a method for identifying steady states of an IRN, namely solving for the global minimizers of $\Phi(w)$. Depending on the decision rule being implemented, local minimizers of $\Phi(w)$ may also be steady-states for an IRN. Also note that by definition, each adaptation of the decision rule must decrease $\Phi(w)$. This implies that as time progresses the network state becomes progressively more desirable.

[2] The MSINR algorithms of [Hicks_04], however, are explicitly shown to be ordinal potential games.

6.3.2 IRN Convergence and Stability Properties

Of course, when an IRN is a potential game, it preserves the convergence properties of potential games. To generalize these results beyond those IRNs which are potential games necessitates the introduction of the following theorems.

Theorem 6.2: *Convergence of closed decision rules for IRNs*
Given IRN, $\langle N,\Omega,\{u_i\},\{d_i\},T\rangle$, with continuous $\Phi(w)$ and compact Ω, all closed decision rules, *d*, converge to a fixed point of *d*.
Proof: Recall that Zangwill's Convergence Theorem A [Zangwill_69] states that the recursion $w^{k+1}\in d(w^k)$ converges to a fixed point of d if the following conditions are met:

(1) All points $\{\omega^k\}$ lie in a compact set;
(2) There exists a continuous function $a:\Omega\rightarrow\mathbb{R}$ such that:
    (a) if ω is not a fixed point, then $a(w')>a(w)\forall w'\in d(w)$ and
    (b) if ω is a fixed point,
then $a(w')\leq a(w)\ \forall w'\in d(w)$;
(3) *d* is closed at ω if ω is not a fixed point of *d*.

For an IRN with continuous $\Phi(w)$, (1) is satisfied when Ω is compact and (2) is satisfied with $a(w)=-\Phi(w)$. Condition (3) is satisfied when the set of possible adaptations is closed for each interference reducing adaptation which was an assumed condition. Thus Zangwill's is satisfied and *d* must converge to a fixed point of *d*.

As we did in Chapter 4, we can also establish a broad range of decision timings that converge by applying weak FIP to the knowledge that a particular decision rule converges.

Theorem 6.3: *Convergence of asynchronous decision rules*
Given cognitive radio network $\langle N,\Omega,\{u_i\},\{d_i\},T\rangle$ with finite Ω for which *d* realizes an IRN for round-robin *T*, then for random and asynchronous T^*, *d* converges to one of the steady states for *d* under round-robin *T*.
Proof:
Paralleling our analysis from Chapter 4, under the assumption of unilateral adaptations, the change in decision timings does not impact the steady-states of the network. Convergence under *T* implies that under T^* and starting from any initial w there exists a sequence of adaptations that terminate in one of the network steady-states. Further this sequence must occur with non-zero probability. Thus $\langle N,\Omega,\{u_i\},\{d_i\},T^*\rangle$ constitutes an absorbing Markov chain where the steady-states of $\langle N,\Omega,\{u_i\},\{d_i\},T\rangle$ are the absorbing

states. Therefore the network converges to one of the steady states for d under round-robin T.

Note that Theorem 6.3 does not imply that $\langle N, \Omega, \{u_i\}, \{d_i\}, T^* \rangle$ remains a strict IRN. Indeed, blips in the monotonic decrease are to be expected for asynchronous timing and may occur for random timing (unless $\langle N, \Omega, \{u_i\}, \{d_i\}, T \rangle$ is also a potential game). However, $\langle N, \Omega, \{u_i\}, \{d_i\}, T^* \rangle$ and $\langle N, \Omega, \{u_i\}, \{d_i\}, T \rangle$ will both converge to the same steady states even if the steady-state distribution may be somewhat altered by the changed timing process.[3] For example Chapter 7 presents an algorithm for DFS in an 802.11h network which is shown to be an IRN under round-robin or random timings but is not an IRN under asynchronous timings. However, the network still converges to a minimizer of $\Phi(w)$. The results of a simulation of the DFS algorithm in an ad-hoc network for asynchronous timings are shown in Figure 6.1 where the channel numbers correspond to the channel appellations assigned to 802.11 in the 5 GHz band. Note that while the network converges, there are two blips where simultaneous adaptations temporarily increase $\Phi(w)$.

[3] As this forms an absorbing Markov chain, the techniques from Chapter 3 could be used to characterize this distribution of steady-states.

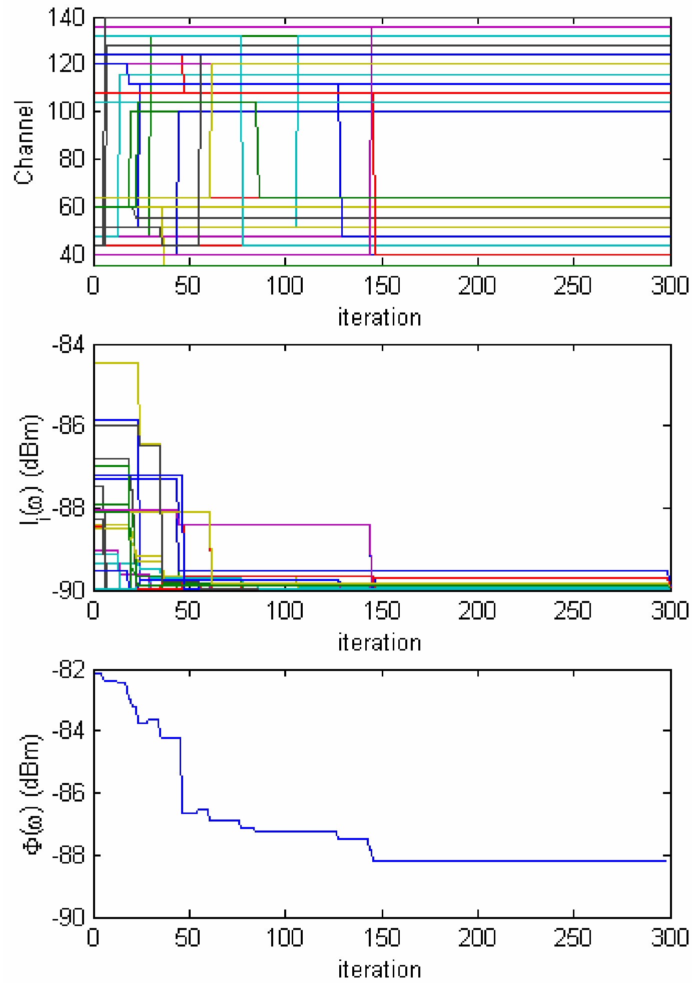

Figure 6.1: Impact of Asynchronous Decision Timings

Likewise, from the definition of a Lyapunov function for discrete time systems [Medio_01], it is immediately apparent that for continuous $\Phi(w)$ and compact Ω, isolated global minimizers of $\Phi(w)$ are Lyapunov stable under the *d* that yields the IRN. Again, isolated local minimizers of $\Phi(w)$ may also be Lyapunov stable for certain network decision rules in an IRN. However, many minimizers of a waveform adaptation process are quite shallow and many are not isolated, so as we show in Section 6.4 stabilizing an IRN frequently requires adjustments to *d* which make *d* an ε-improvement process.

6.4 IRNs that Leverage External Information

It is relatively easy to envision scenarios wherein external observations are leveraged to realize a cognitive radio network that forms an IRN. The following presents two such scenarios.

6.4.1 Globally Altruistic IRNs

Suppose each radio i is capable of measuring its own interference, $I_i(\omega)$, and that this information is distributed to all other cognitive radios in the network. If each radio is guided by the altruistic goal given in (6.4), then this network is an example of a coordination game and thus has exact potential and interference functions given by $V(w) = -\sum_{i \in N} I_i(w) = -\Phi(w)$.

$$u_i(w) = -\sum_{k \in N} I_k(w) \qquad (6.4)$$

Accordingly, if there exists some mechanism such that a single radio adapts at a time (perhaps via a random timer), then all "selfish" adaptations guided by (6.4) result in a monotonically increasing sequence of values for V and a monotonically decreasing sequence of values for Φ and the same steady-states exist and are converged to (though possibly with a different distribution) under asynchronous timing.

In general, a globally altruistic IRN can be achieved by implementing the algorithm given in Algorithm 6.1.

1. Given waveform vector w, select some $i \in N$ to adapt
2. Radio i applies d_i to select some $\hat{w}_i$
3. Radio i polls all $k \in N \backslash i$ for $I_k(\hat{w}_i)$.
4. If (2) increases for i, assign $w = (\hat{w}_i, w_{-i})$, else $w = (w_i, w_{-i})$
5. Return to 1.

Algorithm 6.1: Algorithm for implementing a globally altruistic IRN.

While this approach is likely unsuitable for implementation due to scalability concerns, [Zhao_06] defines one possible means for distributing the global interference measurements throughout the network, namely a radio environment map to which each radio can poll and report observations. Further, several proposed algorithms effectively realize a globally altruistic IRN.

The protocol proposed in [Nie_05], wherein each radio selects from a set of discrete channels in an attempt to minimize network interference, satisfies the conditions to be a globally altruistic IRN. As an interesting note, [Nie_05] proposes that measurements of the interference seen by other radios be estimated at the transmitter by exploiting environmental knowledge – a lower bandwidth, but higher computational complexity approach than in [Zhao_06]. Likewise, as previously covered, the algorithms of [Menon_04], [Popescu_04] and [Sung_03a] also satisfy the conditions of a globally altruistic network.

6.4.2 Locally Altruistic IRNs

Barring the existence of environmental knowledge as in [Nie_05], the amount of information transfer required to support the globally altruistic IRN quickly becomes prohibitive as networks grow in size. However, an IRN can also be created with cognitive radios implementing goals that require the exchange of significantly fewer external observations. Let $I_i \subseteq N$ denote the set of radios with which the signal level of radio i is strong enough to produce non-negligible interference and consider the radio goal shown in (6.5) which considers the interference levels of only those radios in I_i. I_i could be generated using the radio environment map of [Zhao_06] or the available resource map for infrastructure networks proposed in [Krenik_05].

$$u_i(w) = -\sum_{k \in I_i} I_k(w) - I_i(w) \tag{6.5}$$

Interestingly, a network of radios implementing this goal has the same exact potential as the globally altruistic IRN, namely $V(w) = -\sum_{i \in N} I_i(w)$. Accordingly, if there exists some mechanism such that a single radio adapts at a time (perhaps via random backoff), then all "selfish" adaptations guided by (6.5) result in a sequence of monotonically increasing values of $V(w^k)$ and a monotonically decreasing sequence of values of $\Phi(w^k)$ and the same steady-states exist and are converged to (though possibly with a different distribution) under asynchronous timing.

1. Given waveform vector *w*, select some $i \in N$ to adapt
2. Radio *i* applies d_i to select some $\hat{w}_i$
3. Radio *i* polls all $k \in I_i$ for $I_k(\hat{w}_i, w_{-i})$.
4. If (6.5) increases for *i*, assign $w = (\hat{w}_i, w_{-i})$, else $w = (w_i, w_{-i})$
5. Return to 1.

Algorithm 6.2: Algorithm for implementing a locally altruistic IRN.

6.5 IRN Protocols with Internally Generated Observations

While designing cognitive radio algorithms to incorporate external observations permits the network to implement IRNs for a wide variety of network topologies and adaptations, it can introduce significant overhead to the system thereby reducing the spectrum efficiency gains of using cognitive radio. Interestingly, there exist operating scenarios that result in IRNs for broad classes of cognitive radio decision rules that do not rely on external observations. This apparent "free lunch" outcome results from the fact that for certain network topologies and operating scenarios there is an implicit distribution of interference measurements throughout the network via the condition of bilateral symmetric interference.

Definition 6.2: *Bilateral Symmetric Interference*
Two cognitive radios, $j, k \in N$, exhibit bilateral symmetric interference if $g_{jk} p_j r(w_j, w_k) = g_{kj} p_k r(w_k, w_j) \quad \forall w_j \in \Omega_j, \forall w_k \in \Omega_k$ where p_k is the transmission power of radio *k*'s waveform, g_{kj} is the link gain from the transmission source of radio *k*'s signal to the point where radio *j* measures its interference, $r(w_k, w_j)$ represents the fraction of radio *k*'s signal that radio *j* cannot exclude via processing (perhaps via filtering, despreading, or MUD techniques).

In general, $r(w_k, w_j)$ is determined by the absolute value of the correlation between the signal space basis functions modulated by ω_k and ω_j. So we frequently encounter situations where $r(w_k, w_j) = r(w_j, w_k)$, e.g., adjacent and co-channel interference and cross correlation between signature sequences. There are some situations, however, for which $r(w_k, w_j) \neq r(w_j, w_k)$, most notably in typical beam forming applications. Additionally, differences in power or link gain can lead to violations of the bilateral symmetric interference condition. Nonetheless, we are able to identify several scenarios for which the bilateral symmetric interference condition holds. When bilateral symmetric interference holds, any waveform adaptation by radio j that reduces the effective interference that j measures from k's signal results in an equal reduction in effective interference for radio k with respect to j's signal. Thus intending only its own gain, a radio's selfish adaptations nonetheless improves the performance of both radios.

Using the following pair of equations, we can draw a powerful relationship between bilateral symmetric interference and IRNs.

$$u_i(w) = -I_i(w) = -\sum_{k \in N \setminus i} g_{ki} p_k r(w_k, w_i) \tag{6.6}$$

$$V(w) = -\sum_{i \in N} \sum_{k=1}^{i-1} g_{ki} p_k r(w_k, w_i) \tag{6.7}$$

<u>Theorem 6.4</u>: *Bilateral Symmetric Interference and IRNs*

Given $\langle N, A, \{-I_i(w)\}, \{d_i\}, T \rangle$, the following the following three conditions hold, then the network is an IRN

1) N is a network of cognitive radios with adaptations guided by (6.6).
2) All adaptations are unilateral (no more than one radio adapts for each $t \in T$).
3) Bilateral symmetric interference holds for all $j, k \in N$.

Proof: Under the bilateral symmetric interference condition, (6.6) can be equivalently expressed as $u_i(w) = -\sum_{k \in N \setminus i} b_{ki}(w_k, w_i)$ where $b_{ki}(w_k, w_i) = b_{ik}(w_i, w_k)$. Thus these goals are of the form required to be a BSI game (defined in Chapter 5) where $S_i(w_i) = 0 \ \forall i \in N$ so the game has an exact potential given by (6.7).

As a bilateral symmetric interference network is also an exact potential game, all sequences of selfish unilateral deviations increase the value of (6.7). However, note that the interference function for such a network is just $\Phi(w) = -2V(w)$ so any sequence of adaptations that results in a monotonically increasing V also results in a monotonically decreasing Φ, i.e., an IRN.

Again, note that by Theorem 6.3, that adaptations of a finite network will converge to reduced interference states under round-robin or random timing, then it also will under asynchronous timing. The remainder of this section describes three operating scenarios that satisfy the condition of bilateral symmetric interference networks and are thus IRNs for all sequences of unilateral adaptations guided by the goal in (6.6). These scenarios are verified as IRNs via simulations of networks implementing DFS algorithms and spreading code adaptation.

6.5.1 Networks of Isolated Clusters

Encountered in infrastructure based networks employing code or frequency reuse, in a network of isolated clusters the network consists of a set of clusters C for which the following operational assumptions hold:

1. Perhaps through judicious frequency or code reuse between clusters, each radio i is operating in a cluster $c \in C$ for which I_i (the set of radios with which i interferes) is a subset of c.
2. The cluster head enforces a uniform receive power, r_c, on all radios k for signals transmitted to the cluster head.
3. Waveforms are restricted to those waveforms for which $r(w_k, w_i) = r(w_i, w_k)$.
4. Cluster heads provide measurements of (6.6) to all client radios in the cluster.

Under these assumptions it is readily apparent that the bilateral symmetric interference condition holds [4], thereby ensuring that all sequences of unilateral self-interested adaptations realize an IRN.

Though formulated differently and generally considering specific decision rules instead of any selfish decision rule, such a network is utilized in [Hicks_04], [Ulukus_01a], [Rose_02], [Ulukus_01b], [Popescu_02], [Concha_01] for spreading code adaptations. An algorithm for implementing a network of isolated clusters IRN in each cluster c is given in Algorithm 6.3.

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[4] Symmetric received power implies $g_i p_i = g_k p_k$. Coupled with the condition that $r(w_k, w_i) = r(w_i, w_k)$, it is clear that $g_i p_i r(w_k, w_i) = g_k p_k r(w_i, w_k)$.

1. Given waveform vector w, select some $i \in c$ to adapt.
2. Radio i applies d_i to select some $\hat{w}_i$
3. The cluster head for c returns the updated value given by (4) to i.
4. If (4) increases for i, assign $w = (\hat{w}_i, w_{-i})$, else $w = (w_i, w_{-i})$
5. Return to 1.

Algorithm 6.3: Algorithm for implementing an IRN for a network of isolated clusters.

Verifying that such a network constitutes an IRN, the results of a simulation of seven code-adapting (over six signal-space dimensions) cognitive radios guided by (6.6) and communicating at a constant received power (-50 dBm) with a common cluster head is shown in Figure 6.2. The top plot shows the measured interference levels for the each of the cognitive client radio and the bottom plot shows $\Phi(\omega)$ for the network. Note that each adaptation of the network reduces the value of $\Phi(\omega)$ thereby satisfying the defining condition of an IRN.

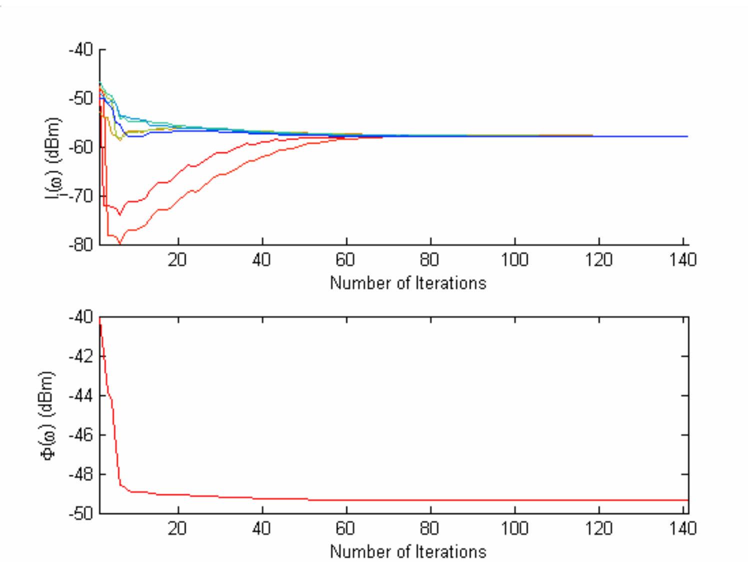

Figure 6.2: Simulation of seven code adapting cognitive radios operating in an isolated cluster. [Neel_06a]

6.5.2 Close Proximity Networks

In this operational scenario it is assumed that the radios are operating as an ad-hoc network in sufficiently close proximity and transmitting with sufficiently similar power levels that waveform correlation dominates the distance and transmitted power effects are negligible. Such a scenario may arise in a network of closely spaced WLAN devices where the presence of any in-band energy triggers a collision event, although such a network would constitute an *effective* interference reducing network as opposed to the strict interference reducing network. Under these assumptions (6.6) is equivalent to (6.8).

$$u_i(w) = -\sum_{k \in N \setminus i} r(w_k, w_i) \tag{6.8}$$

If we again assume that $r(w_k, w_i) = r(w_i, w_k)$, then the system satisfies the bilateral symmetric interference condition and thus forms an IRN for all sequences of unilateral self-interested unilateral adaptations. An example of an algorithm that could be implemented to realize an IRN in a close proximity network is given in Algorithm 6.4.

1. Given waveform vector w, select some $i \in N$ to adapt.
2. Radio i applies d_i to select some $\hat{w}_i$
3. If (6.8) increases, assign $w = (\hat{w}_i, w_{-i})$, else $w = (w_i, w_{-i})$
4. Return to 1.

Algorithm 6.4: Algorithm for implementing an IRN for a close proximity network.

Note that many implementations can improve upon this algorithm by exploiting information specific to the waveform being adapted. Such a situation is considered in the example of DFS in a close proximity network in an ad-hoc DFS network we presented in [Neel_05]. In that example, the cognitive radios are formed into a closely-spaced ad-hoc network with each cognitive radio evaluating FFTs to identify the frequency at which signal spacing is maximized. The goals in that example can be equivalently reformulated into the IRN form given in (6.8) as shown by (6.9) where $s(f_i, f_k) = \max\{B - |f_i - f_k|, 0\}$, f_i is the frequency of radio i, f is the frequency vector determined by the choices of frequency by all radios, and B is the bandwidth of the signals. Intuitively, this goal expresses a preference for greater frequency spacing up to the point where no signal overlap occurs.

$$u_i(f) = -\sum_{k \in N \setminus i} s(f_i, f_k) \tag{6.9}$$

Note that as formulated, $s\left(f_i, f_k\right) = s\left(f_k, f_i\right)$, thereby satisfying bilateral symmetric interference. Thus it is expected that any sequence of self-interested unilateral frequency adaptations guided by (6.9) will behave as an interference reducing network.

The operation of such a DFS network can be visualized via the following example simulation of ten cognitive radios which are operating in close proximity, free to adapt over a policy determined 10 MHz of available center frequencies, and supporting applications that require with 1 MHz bandwidth signals. The results of a simulation this network where a randomly selected radio is permitted to adapt at each iteration are depicted in Figure 6.3 and Figure 6.4. The top plot depicts the operating frequencies of each radio starting from random initial distributions of frequencies; the middle plot shows the evaluation of the goals of all the radios in the network; the bottom plot shows the value of $\Phi(w)$.

While implementing the same decision rule (choosing to operating at the frequency that minimizes interference) in both simulations, different steady-states are reached because of different initial states for the network. However, in both simulations $\Phi(w)$ forms a bounded monotonically decreasing sequence and the adaptations reach a steady-state condition from which no further unilateral adaptations can improve a radio's goal. Also note that while the second simulation in Figure 6.4 does achieve a globally optimal solution, i.e., $\Phi(\omega)=0$, this is not a property guaranteed to hold for IRNs as shown in the simulation of Figure 6.3 which achieves a good, though non-optimal solution.

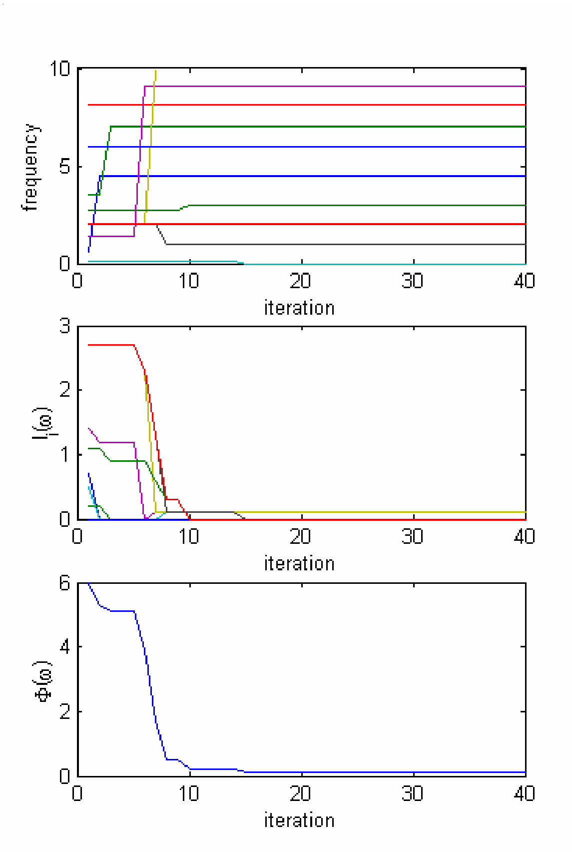

Figure 6.3: Simulation of a close proximity network.

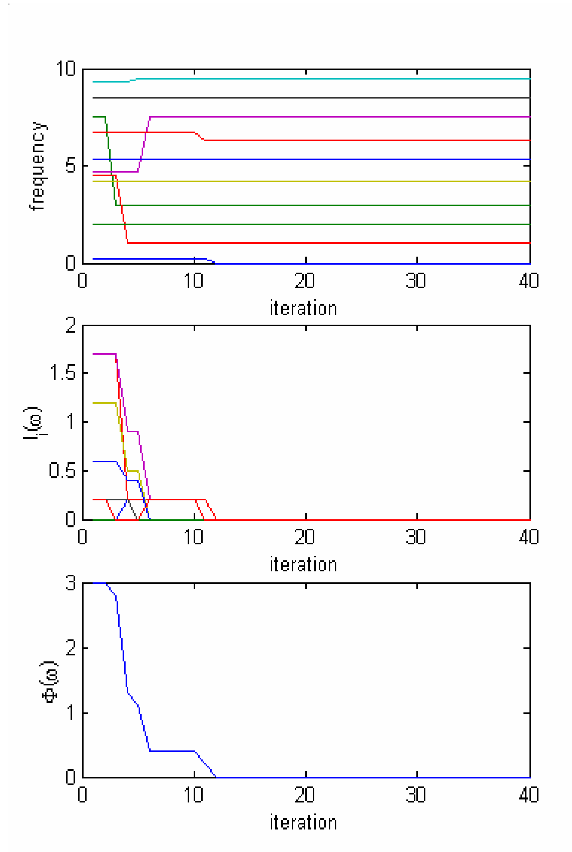

Figure 6.4: Simulation of a close proximity network.

6.5.3 Controlled Observation Processes

Instead of adjusting network topologies, it is also possible to achieve the bilateral symmetric interference condition by controlling the observation processes of the cognitive radios.

For instance suppose a network consists of a collection of clusters where each cluster is controlled by a fixed access node whose adaptations are guided by the goal given in (6.6). To estimate interference, each access node measures the received signal power of the RTS/CTS signals in the network which are assumed to be broadcast at the same maximum transmit power level, p. Further it is assumed that $g_{jk} = g_{kj}$ [5] and waveforms

[5] It is permissible that link gains between access nodes are frequency selective, but frequency selectivity of the gains must be reciprocal as well, i.e., $g_{ik}(f_i) = g_{ki}(f_i)$. For purposes of analysis, this frequency selectivity can be considered a part of $r(w_i, w_k)$.

are restricted to those waveforms where $r(w_k, w_i) = r(w_i, w_k)$. As a result of these restrictions on the observation processes, it is readily verified that $g_{kj} p_k r(w_k, w_j) = g_{jk} p_j r(w_j, w_k) \;\; \forall j,k \in N$ thereby satisfying the bilateral symmetric interference condition. Such a network could be encountered in a business WLAN installation where multiple access nodes with the same maximum transmit power level are deployed throughout a building or in an infrastructure based WRAN deployment, thereby ensuring that the access nodes all have the same maximum transmit power. An example of an algorithm that could be implemented to realize an IRN in a close proximity network is given in Algorithm 6.5. Again, depending on Ω and the network topology it may be possible to design d_i such that adaptations will be known to increase (6.6) without needing to specifically evaluate (6.6) post-adaptation.

1. Given waveform vector *w*, select some $i \in N$ that will be allowed to adapt.
2. Radio *i* applies d_i to select some $\hat{w}_i$
3. If (6.6) increases assign $w = (\hat{w}_i, w_{-i})$, else $w = (w_i, w_{-i})$
4. Return to 1.

Algorithm 6.5 Algorithm for implementing an IRN under a controlled observation process.

A simulation of thirty access nodes randomly distributed over 1 km^2 with a path loss exponent of 3 implementing DFS with random adaptation timings and guided by (6.6) is shown in Figs. 9 and 10 starting from two different random distributions of frequencies. Note that while the networks converge to different steady-state frequency distributions, in both cases, $\Phi(\omega)$ forms a monotonically decreasing sequence. Also note that unlike before, these frequencies are not approximately regularly spaced as before. Instead the network has naturally converged to a frequency reuse scheme wherein sufficiently separated radios operate at the same, or nearly the same, frequency.

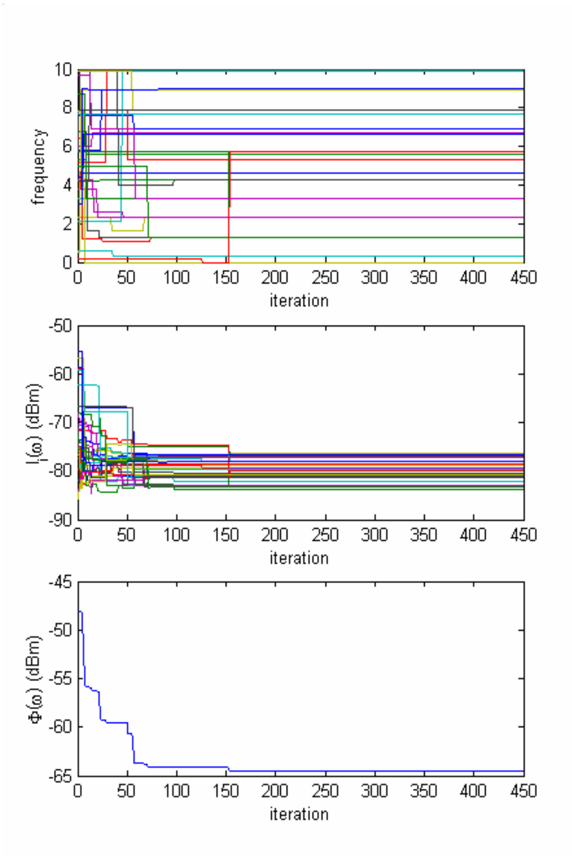

Figure 6.5 30 randomly distributed DFS nodes adapting to interference measured at the transmitter.

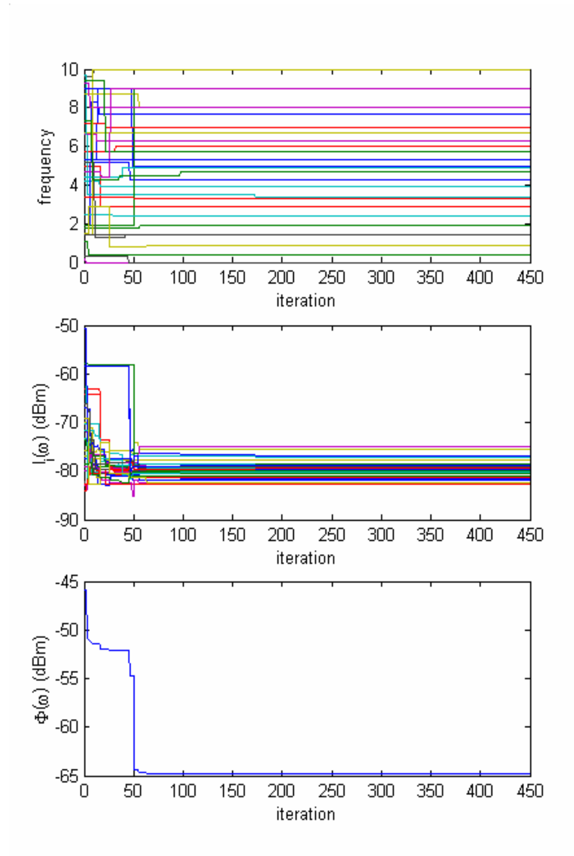

Figure 6.6 Simulation of system in Figure 6.5 with different initial frequencies.

6.6 Stabilizing IRNs

Recall that IRNs are guaranteed to converge to a steady state but that many steady-states are not guaranteed to be stable because they are not isolated. This effect can be seen in Figure 6.7 where interference estimations are corrupted by noise at -90 dBm. While the networks again achieve reasonably good performance, the radios in the network are almost continuously adapting their transmit frequencies which reduce the intended gains in spectral efficiency. This phenomenon can be limited by adjusting the decision rule d_i such that a radio only adapts its transmit frequency if the resulting frequency is predicted to improve performance by at least some threshold, $\tau>0$, or generalizing to waveform adaptations as shown in (6.10).

$$\tilde{d}_i(w)=\begin{cases} d_i(w) & u_i(d_i(w),w_{-i})\geq u_i(w)+t \\ w_i & otherwise \end{cases} \tag{6.10}$$

Note that this is an example of an ε-better response decision rule which converges for finite and infinite spaces under round-robin, random, and asynchronous timings for exact potential games with bounded continuous potential functions on a compact action space. With this threshold included, the system becomes stable as shown in Figure 6.8. However, this threshold slightly increases the operating interference levels of the network as the network is now stops adapting when it reaches an ε-NE.

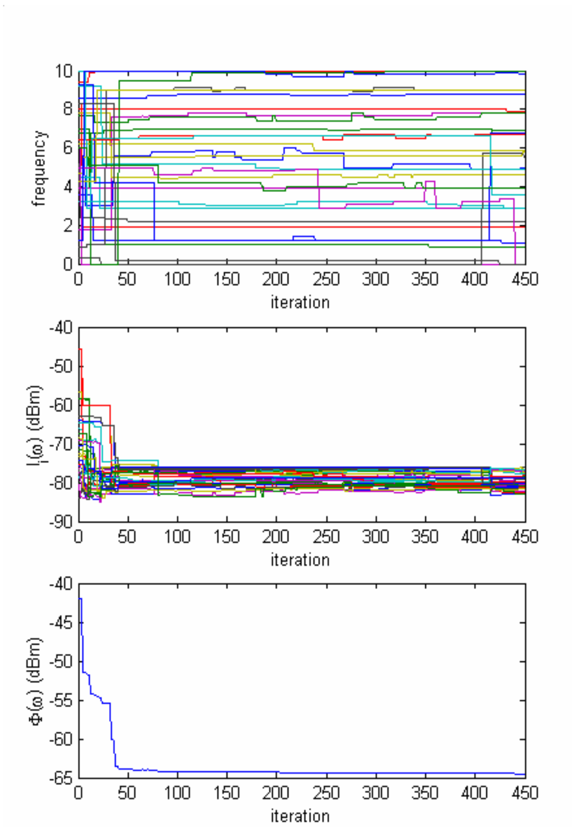

Figure 6.7 Simulation of system in Figure 6.5 where interference estimates are corrupted by noise.

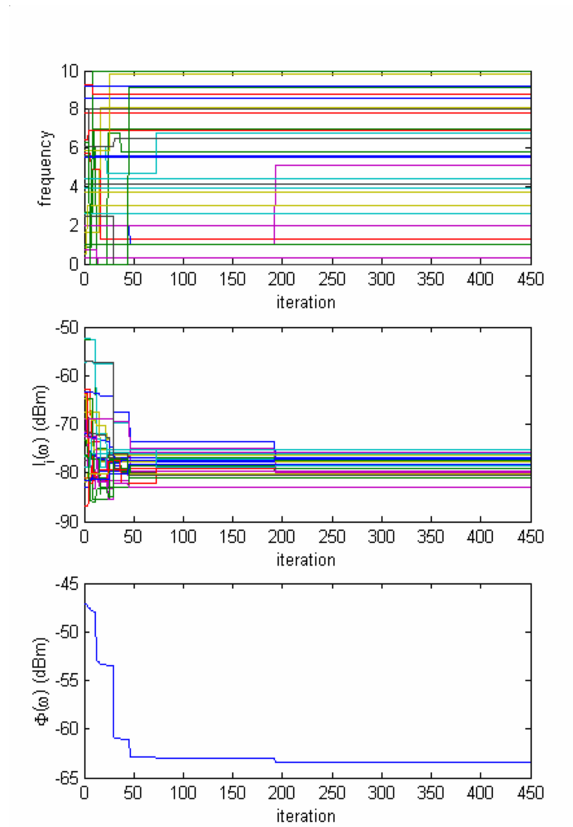

Figure 6.8 Simulation of system in Figure 6.5 where interference estimates are corrupted by noise, but threshold adaptations are employed.

A similar scenario could be readily constructed for pure ad-hoc networks that observe RTS-CTS signals to estimate (6.6). Such a scenario may arise in an infrastructureless WRAN (802.22) or in an 802.11 ad-hoc network. The results from a simulation of an ad-hoc code adapting network of twelve cognitive radios sharing six signal dimensions randomly distributed over 2500 m^2 with a path loss exponent of 3 is shown in Figure 6.9 and Figure 6.10. As was the case for the DFS system, adaptations still rapidly decrease the value of Φ(ω) until the network nears a steady-state at which point the presence of noise can lead to an excessive number of adaptation as implied by the noisy simulation shown in Figure 6.9. However, the addition of a requirement that selfish adaptations decrease interference by at least a small threshold can again stabilize the network as indicated by Figure 6.10. Further it provides a means to estimate convergence time as shown in Theorem 6.5 which applies the convergence rate theory of Chapter 5 to IRNs.

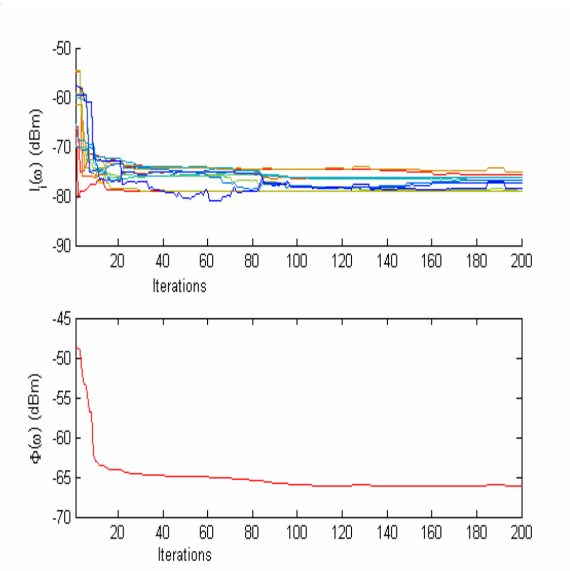

Figure 6.9: Code adaptation in a noisy ad-hoc network.

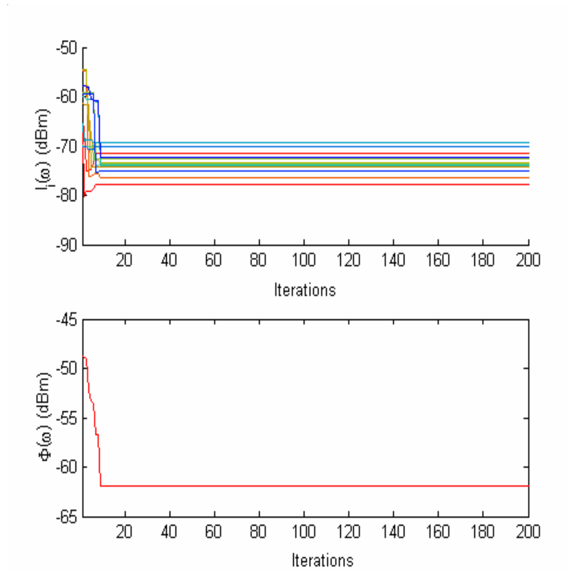

Figure 6.10: Code adaptation in a noisy ad-hoc network where adaptations only occur if interference decreases interference by at least τ.

Theorem 6.5: *Convergence time of threshold adaptations*
Given an IRN with continuous exact potential *V* and compact Ω where Φ = -α*V*, $a \in \mathbb{R}^+$, if *d* is modified to be of the form given in (8), then no more than $\left[\Phi(w)-\Phi^{\min}(w)\right]/(a t)$ adaptations can be taken before arriving at a steady-state where Φ(ω) is the current interference level and $\Phi^{\min}(\omega)$ is the global minimum of Φ.
Proof: Theorem 6.1 supplies the existence of $\Phi^{\min}(\omega)$. By the definition of an exact potential game, each adaptation that increases u_i by τ increases *V* by τ and decreases Φ by ατ. Dividing the difference between Φ(ω) and $\Phi^{\min}(\omega)$ by ατ yields the desired expression.

6.7 Legacy Devices and IRNs

The implicit observation and cooperation inherent to the bilateral symmetric interference condition extends to legacy devices which are unable to adapt their waveforms. If we let

L denote the set of legacy radios in the network where it is assumed that $L \cap N = \varnothing$, then (6.11) gives a goal for a cognitive radio seeking to decrease its own interference levels.

$$u_i(w) = -\sum_{k \in N \setminus i} g_{ki} p_k r(w_k, w_i) - \sum_{k \in L} g_{ki} p_k r(w_k, w_i) \tag{6.11}$$

Under the assumption that the left portion of (6.11) satisfies the bilateral symmetric interference condition, it is relatively straight forward to establish the following conditions for which the entire goal given in (6.11) will also satisfy bilateral symmetric interference.

- *Networks of Isolated Clusters* – Trivially, if the legacy devices are also operating in their own isolated clusters, then $g_{ki} p_k r(w_k, w_i) = g_{ik} p_i r(w_i, w_k) = 0 \quad \forall i \in N, k \in L$.
- *Close Proximity Networks* – If it is assumed that the legacy devices are also operating in close proximity and $r(w_k, w_i) = r(w_i, w_k)$, then $g_{ki} p_k r(w_k, w_i) = g_{ik} p_i r(w_i, w_k)$.
- *Controlled Observation Processes* – If $\forall i \in N$ the observation processes form a controlled observation process and it is assumed that the legacy devices have the same transmit power level, then $g_{ki} p_k r(w_k, w_i) = g_{ik} p_i r(w_i, w_k)$.

Under all three scenarios, an exact potential exists and is given by $V(w) = -\sum_{i \in N \cup L} \sum_{k=1}^{i-1} g_{ki} p_k r(w_k, w_i)$ which is related to Φ(ω) as Φ(ω)=-2*V*(ω) implying that the network is again an IRN even with legacy devices included in the network.

Modifying the earlier simulation of 30 randomly distributed controlled-observation DFS devices so that five devices are incapable of adaptation, Figure 6.11 and Figure 6.12 show the results of the simulation. Note that the interference seen by both the legacy (dotted lines) and the cognitive radios (solid lines) decrease during the operation of this network. However, the legacy radios tend not to achieve as low of an interference level, primarily because they are unable to separate themselves in frequency from other legacy radios in the network.

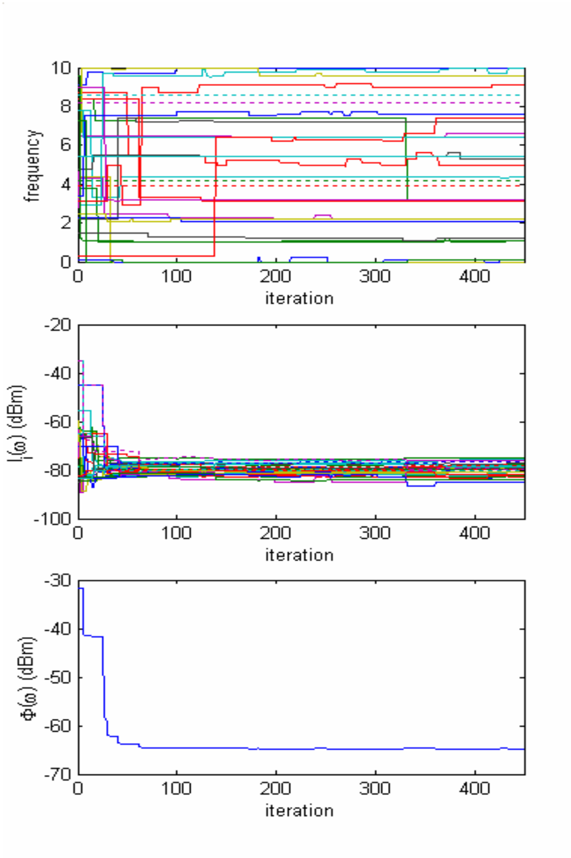

Figure 6.11: Noisy simulation of system in Figure 6.5 where five devices are incapable of adapting.

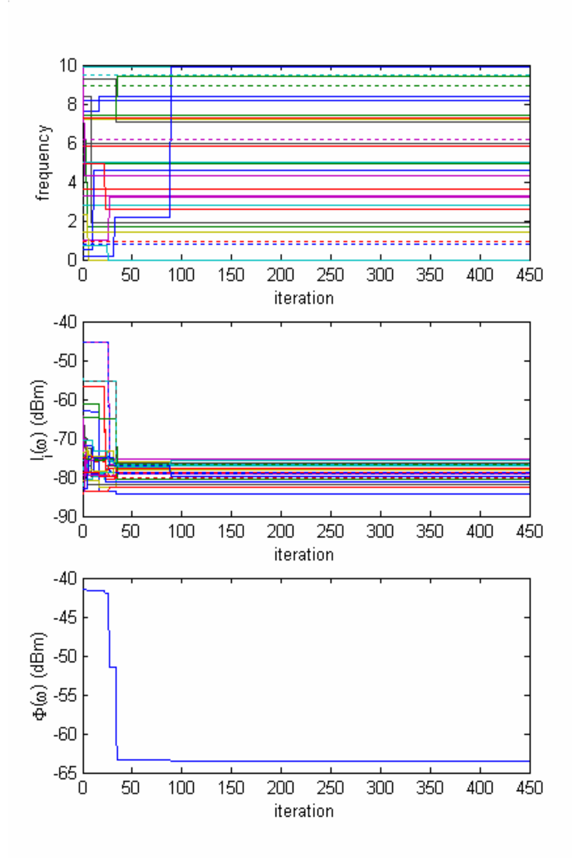

Figure 6.12: Noisy simulation of system in Figure 6.5 where five devices are incapable of adapting, and the rest implement a threshold adaptation.

It is also interesting to note that the continued existence of an exact potential game is not impacted if the addition of legacy radios violates the bilateral symmetric interference condition, perhaps by transmitting at a different power level. In such a situation, each legacy interference component forms a self-motivated function discussed in Chapter 5 and takes on the form shown in (6.12).

$$S_i(w_i) = \sum_{k \in L} g_{ki} p_k r(w_k, w_i) \quad (6.12)$$

Combining this with the original utility function yields the expression shown in (6.13) which is the more general form of a BSI game defined in Chapter 5.

$$u_i(w) = -\sum_{k \in N \setminus i} g_{ki} p_k r(w_k, w_i) - \sum_{k \in L} g_{ki} p_k r(w_k, w_i) \quad (6.13)$$

Applying the potential function given in Chapter 5 for BSI games yields an exact potential given by (6.14).

$$V(w) = -\sum_{i\in N}\sum_{k=1}^{i-1} g_{ki} p_k r(w_k, w_i) - \sum_{i\in N}\sum_{k\in L} g_{ki} p_k r(w_k, w_i) \qquad (6.14)$$

However, the relation Φ(ω)=-2V(ω) no longer holds. Under these conditions, sequences of unilateral self-interested adaptations will continue to monotonically increase V and decrease the sum interference perceived by the cognitive radios but they are not guaranteed to monotonically decrease Φ as summed over $N \cup L$.

6.8 IRN Summary and Conclusions

This chapter has proposed a powerful new framework for the design of cognitive radio algorithms – the interference reducing network – for which adaptations converge and for which each adaptation improves network performance. For arbitrary conditions, this policy can be implemented by incorporating observations made by other cognitive radios into altruistic goals of the cognitive radios. However, when the bilateral symmetric interference condition holds, the radios only need to utilize estimates of their own interference to inform their decision making processes resulting in networks with excellent performance and minimal overhead (presumably some signaling is required to support the adaptation of a link, but there is no need to distribute additional information to coordinate these decisions with other links in the network).

However, these internal information scenarios are generally limited to those waveforms for which the reciprocal energy property holds. For example, it is relatively easy to envision transmit beam forming adaptations that while operating in one of the identified internal information scenarios and implementing locally desirable adaptations would nonetheless not implement an IRN. Conversely, receive beam forming adaptations would also not satisfy the reciprocal energy property, yet will implement an IRN for any network topology. [6] Noting the fundamental tradeoff between external and internal

[6] Consider any network topology implementing receive beam forming. Each self-interested adaptation guided by reducing received interference will presumably decrease the interference measured at the adapting device while having no impact on the other cognitive radios in the network. Thus, given a sequence of such adaptations, the sum of all measured interference levels, Φ(ω), forms a monotonically decreasing sequence.

observations, namely complexity versus generalizability, an interesting line of research becomes immediately apparent – how can cognitive radios recognize when they are operating under the bilateral symmetric interference condition so that higher efficiency networks might be realized? This is a topic beyond the scope of this document, but is the intended subject of [Neel_06b].

For each scenario considered in this chapter, an exact potential game model was identified. While the existence of an exact potential game model is useful for analysis, it holds further implications for ontologically reasoning cognitive radios, that is to say, radios for which decision processes cannot be determined *a priori*, and are instead learning as they go. In such an event, the existence of an exact potential provides an assurance that as long as adaptations are performed unilaterally and each ontologically-defined radio acts to improve its own performance the network will constitute an IRN. Likewise, rather than the specific procedures followed in calculations of [Ulukus_01a], [Rose_02], [Ulukus_01b], [Popescu_02], and [Concha_01], the radios can choose any waveform that improves its own goal. Further, the controlled observations scenario permits us to implement these algorithms without having to have a common receiver (or collocated receivers) as in [Hicks_04], [Menon_04], [Ulukus_01a], [Rose_02], [Ulukus_01b], [Popescu_02], and [Concha_01]. Finally the BSI condition assures us that this can be performed without the collaboration of other radios as in [Popescu_04], [Sung_03a], and [Nie_05].

The examples considered in this chapter should represent only a fraction of cognitive radio networks that could be designed to implement IRNs and only a fraction that satisfy the bilateral symmetric interference condition. It should be possible to identify additional IRNs by considering alternate topology and observation constraints, adaptations beyond frequency and spreading codes, and combinations of constraints and multiple adaptable waveform parameters.

6.9 References

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Chapter 7: Dynamic Frequency Selection[1]

"So Abram said to Lot, "Let's not have any quarreling between you and me, or between your herdsmen and mine, for we are brothers.
Is not the whole land before you? Let's part company.
If you go to the left, I'll go to the right; if you go to the right, I'll go to the left."
- Genesis 13:8-9

Over the past years the number of WiFi hotspots has exploded. We all know that if we go downtown or to a large apartment building, we can find dozens of open access points to log on to. In fact Philadelphia, San Francisco, and our own Roanoke have rolled out city-wide WiFi services. So even before the wide-scale deployments of 3G and WiMAX systems high speed data services are already ubiquitous.

While WiFi coverage has become less of a problem, external network interference has emerged as a significant problem as the networks fight for access to a limited number of channels (and frequently, the same channel!). In theory, this interference problem could be ameliorated by applying a frequency reuse pattern to the networks – a seemingly easily implemented approach as 802.11b has three nonoverlapping channels (1, 6, and 11) and 802.11a has eight minimally interfering channels in the US (nineteen in Europe) which are explicitly intended to facilitate frequency reuse in a minimally interfering manner. However, most people never modify their access points from the factory settings so many access points operate on the same pre-set channel.[2]

A few years ago this reliance on the factory settings became a problem for a friend of mine he set up a WiFi network in his house. A few months after setting up his network, his neighbors set up their own WiFi networks in their houses. As everyone had left the access points with their factory settings and had bought the same model of access point,

[1] This chapter is mostly taken from [Neel_06].

[2] An online acquaintance of mine has noted this same phenomenon with compasses in cars. Because magnetic north is not true north, the direction a compass points varies geographically. To combat this, compasses in cars intended for the US come equipped with fifteen different settings which calibrate the difference between magnetic north and true north. Invariably when he has bought a car, whether new or used, whether in California or in Virginia, the car's compass has been set for region 8 – Detroit, the factory site. So right now in the US, there's thousands of people who think they're driving north when they're really driving northwest or northeast because they never changed their compass from its default setting.

all of the networks were operating on channel 6 and the performance of my friend's network suffered noticeably from the interference. Being a PhD wireless engineer at Virginia Tech which confers a certain degree of mischievousness, my friend was not content to simply reconfigure his access point to operate on channel 1 or 11. At the time leaving an access point on its factory settings meant that the access point had no security and a common password, so he logged into his neighbors' access points and changed their operating channels so that all three access points were operating on non-interfering channels – a process he extended to his entire street as new access points were added. In this way, an optimal frequency reuse scheme was found for everyone's networks even if most parties were unwitting participants in the process.

Unfortunately when designing multi-channel networks, we can not count on the networks being deployed on streets with wireless engineers willing to tune the networks so another solution is required. Instead, we would prefer to construct self-tuning networks wherein the networks autonomously choose their parameters post-deployment indicating an opportunity for one of the killer applications of cognitive radio identified in Chapter 1 – automated radio resource management for automated deployment. Ideally, we would like for channel tuning portion to be as effective and as simple as possible – perhaps as simple as when Abram ensured that his flock would not interfere with Lot's with the promise, "*If you go to the left, I'll go to the right; if you go to the right, I'll go to the left.*"

Leveraging the insights gained over the preceding chapters, this chapter proposes low-complexity algorithms for autonomous dynamic frequency selection (DFS) for interference minimization among secondary users[3]. These algorithm are suitable for implementation in 802.22 and 802.11h networks, require no direct collaboration between devices and are easily implemented. Section 7.1 introduces a general model of DFS adaptation algorithms and presents related work. Section 7.2 formally introduces the algorithms, proves important results related to steady-states, the desirability of those steady-states, convergence, and stability and verifies these results via simulation. Section

[3] Dynamic frequency selection is proposed in 802.11h and 802.22 primarily as a means to minimize interference with primary spectrum users – military radars and television broadcasts, respectively – with minimization of interference to other 802.11h and 802.22 devices a secondary consideration.

7.3 studies the proposed algorithms under various realistic conditions – policy variations, asynchronous timing, local frequency preference, noise, and the impact of differing power levels.

7.1 Background

This section introduces a model of distributed DFS algorithms, presents related work, and briefly reviews the IRN design framework.

7.1.1 Modeling DFS Algorithms

Modifying the notation of Chapter 2 to be DFS specific, we can model a network of cognitive radios (or any goal-driven adaptive radios) by the tuple, $\langle N, F, \{u_i\}, \{d_i\}, T\rangle$ where N represents the set of n cognitive radios, F is the frequency space formed as $F = F_1 \times \cdots \times F_n$ where F_i specifies the frequencies available to cognitive radio $i \in N$, $\{u_i\}$, $u_i : F \rightarrow \mathbb{R}$, is the set of goals that inform the cognitive radios' decision processes, $d_i : F \rightarrow F_i$, which are implemented at the decision timings contained in T.

For our DFS algorithm we model the goal of the radios as minimizing perceived interference as shown in (7.1)

$$u_i(f) = -I_i(f) = -\sum_{k \in N \setminus i} g_{ki} p_k s(f_i, f_k) \tag{7.1}$$

where σ measures the fractional interference, i.e., $s(f_i, f_k) = \max\{B - |f_i - f_k|, 0\}/B$, f_i is the frequency of cognitive radio i's RTS/CTS signal, p_k is the transmission power of radio k's waveform, and g_{ki} is the link gain from the transmission source of radio k's signal to the point where radio i measures its interference. It is assumed that the network design objective function is to minimize the sum network interference, $\Phi(f)$, as shown in (7.2).

$$\Phi(f) = \sum_{i \in N} \sum_{k \in N \setminus i} g_{ki} p_k s(f_k, f_i) \tag{7.2}$$

7.1.2 Related Work

Many authors have attacked the problem of DFS, or more generally dynamic spectrum access (DSA), by requiring assuming a centralized decision maker. After noting that finding the optimal frequency allocation is a NP-complete problem, [Leung_03] proposes a heuristic centralized algorithm based on a local search algorithm with random restart to search through the possible frequency combinations. As part of a solution to network formation problem [Steenstrup_05] utilizes a central controller to assign frequencies to each link in the network according to the abbreviated algorithm shown in Figure 7.1.

For each $v_i \in V$ do
  $F_i^R \leftarrow F_i - P_i$
  If $F_i^R = \emptyset$, no feasible assignment exists
  $F_i^T \leftarrow F_i^R$
For each $v_i \in V$ do
  For each v_j s.t. $(v_i, v_j) \in E$ do
    $F_i^T \leftarrow F_i^T \cap F_j^R$
For each $V_i \in V$ do
  $B \leftarrow \emptyset$ (indices of boundary nodes)
  For each v_j s.t. $(v_i, v_j) \in E$ do
    If $F_i^T \cap F_j^T = \emptyset$
      If $F_i^T \cap F_j^R \neq \emptyset$ and $F_j^T \cap F_i^R \neq \emptyset$
        $B \leftarrow B \cup \{i\} \cup \{j\}$
      Else, no feasible assignment exists
For each $b \in B$ do
  $S \leftarrow \{v_b\}$ (nodes with common broadcast frequencies)
  $C \leftarrow F_b^T$ (common broadcast frequencies)
  $U \leftarrow \{v_b\}$
  For each $v_i \in U$ do
    For each v_j s.t. $(v_i, v_j) \in E$ do
      If $F_i^T \cap F_j^T \neq \emptyset$
        $S \leftarrow S \cup \{v_j\}$
        $C \leftarrow C \cap F_j^T$
        $U \leftarrow U \cup \{v_j\}$
        If $v_j \in B$
          $B \leftarrow B - \{v_j\}$
    $U \leftarrow U - \{v_i\}$
  $c \leftarrow$ any frequency in C
  For each $v_i \in S$ do
    $c_i \leftarrow c$ (selected broadcast frequency for v_i)

Figure 7.1: Frequency Assignment Algorithm in [Steenstrup_05]

Other authors do not assume a central controller, but instead assume extensive message passing between the devices so each radio can effectively calculate the same solution.
For instance, [Zhao_05] considers a network of orthogonal channels where adaptive secondary users coordinate their adaptations via a common channel. [Etkin_05] considers a system wherein optimal frequency/power allocations are achieved by employing punishment strategies similar to the ones considered in Chapter 4 but applied to DFS

where the optimal strategy is known *a priori.* [Nie_05] considers a DSA scheme wherein radios must share information over a common channel to compute the interference levels each radio would induce to other radios in order to evaluate its goal (U2 in [5]). While this has the virtue of being both an exact potential game and an IRN, it requires significant overhead to distribute the information needed to evaluate the goal and requires that decisions are made sequentially. For DSA systems where spreading codes adapted (viewed in the context of signal space representations, spreading code adaptation algorithms could be directly applied to DFS problems), [Sung_03a] presents an algorithm where each radio's goal incorporates the interference measurements of all other radios in the system. [Xing_06] considers a *Homo Egualis* ("fair man") implementation where each access point chooses frequencies so as to maximize (7.3) where x_j is the usable spectrum for user j and $a_i, b_i \in \mathbb{R}$. Thus each access point attempts to ensure that every access point is receiving approximately the same amount of spectrum.

$$u_i = x_i - \frac{a_i}{n-1}\sum_{x_j > x_i}\left(x_j - x_i\right) - \frac{b_i}{n-1}\sum_{x_j < x_i}\left(x_i - x_j\right) \tag{7.3}$$

[Villegas_05] considers a distributed graph coloring algorithm where edges are formed between interfering access points. Each access node recursively distributes frequency and interference measurements and selects the frequency it believes will result in the least interference.

Other authors have considered single cell adaptations without the need for communication beyond reporting measurements from a common receiver. As discussed in Chapter 6, [Sung_03b], [Hicks_04], and [Ulukus_04] consider spreading code adaptations where each access node is isolated in frequency and spreading codes are chosen so as to minimize the interference of clients/mobiles are – a situation analogous in signal space to DFS applied to the clients in a single isolated cluster.

[Nie_05] also proposes another goal (or utility function) for DSA (U1) that is identical to the goal used in this paper (equation (7.1)). However, because [Nie_5] places no restrictions on the observation mechanism, [Nie_05] is unable to show that system forms an exact potential game which would permit the use of a simple distributed and autonomous algorithm. Instead [Nie_05] employs a no-regret learning algorithm wherein the radios autonomously try every possible frequency and then adapt to frequencies that yield the best weighted cumulative utility and show that the algorithm converges to a mixed-strategy equilibrium – a less than optimal result as mixed strategies in frequency selection imply continuous probabilistic adaptation.

[Luo_04] considers a closely related algorithm applied to a regular 10x10 grid of access points where each radio is guided by (7.4)

$$u_i(f_k) = \frac{M_i}{\sum_{M_j \in S_k} M_j} f\left(\sum_{M_j \in S_k} M_j\right) \tag{7.4}$$

where M_i is the number of users attached to access node *i*, S_k is the set of nodes operating on f_k and *f* evaluates the throughput if for the users in the argument. Each access node then chooses the channel that maximizes its throughput and switches to it with a random probability.

does not have to infer what other radios are experiencing, does not does not assume the existence of a centralized decision maker proposes a low-complexity autonomous distributed DFS algorithm suitable for use in ad-hoc 802.11h networks. After briefly defining the concepts of interference reducing networks and exact potential games and defining the proposed algorithm, this paper shows via analysis and simulation that this algorithm results in a frequency allocation that is a minimizer of sum network interference even when different policies are applied to different channels, asynchronous decision timings are used, access nodes exhibit private frequency preferences, and spectral signals are imperfectly estimated. Additionally, the impact of combining transmit power control (TPC) with our DFS algorithm is explored.

7.1.3 Interference Reducing Networks

Then a cognitive radio network is said to be an *interference reducing network* (IRN) if all adaptations decrease the value of the sum of observed interference levels $\Phi(f) = \sum_{i \in N} I_i(f)$

where $I_i(f)$ is the interference observed by cognitive radio i when the frequency vector $f \in F$ is implemented by N.

Chapter 6 states that an IRN can be realized in a distributed and autonomous fashion by selfish interference minimizing radios if adaptations are made by only one radio at a time under if the condition of *bilateral symmetric interference* (BSI) holds which happens if $g_{ki} p_k s(f_i, f_k) = g_{ik} p_i s(f_k, f_i) \forall f_k \in F_k, \forall f_i \in F_i$. BSI implies that a network is an IRN for unilateral adaptations because BSI implies that is an *exact potential game*. An exact potential game is a normal form game for which there exists a function, called the *exact potential function*, $V: F \rightarrow \mathbb{R}$, such that $u_i(\hat{f}_i, f_{-i}) - u_i(f_i, f_{-i}) = V(\hat{f}_i, f_{-i}) - V(f_i, f_{-i}) \forall i \in N$ where f_{-i} refers to the n-1 dimensional vector formed by excluding the contribution of i from f. By examining this definition, it is apparent that when profitable unilateral adaptations are made in an exact potential game, V constitutes a monotonically increasing sequence. Since when BSI holds, $\Phi(f)=-2V(f)$ Chapter 6, a monotonically increasing V implies a monotonically decreasing $\Phi(f)$ and an IRN is realized. This monotonicity property can then be used to prove the convergence of all selfish decision rules with unilateral timings Chapter 6.

7.2 An IRN DFS Algorithm

As opposed to algorithms with a centralized decision maker, or a single cell network, or a network that requires significant message passing just for the radios to evaluate their own goal, this chapter presents a DFS algorithm proposed in [Neel_06] which is completely distributed and requires no message passing between clusters (presumably an access point has to signal its users when changing frequencies). Further it does this without requiring complex computations or observations – the radio merely has to measure the received power of the RTS/CTS messages sent by other access nodes in the network and then choose a channel that the radio believes will reduce its interference.

This simple algorithm works because it satisfies the Interference Reducing Network (IRN) framework, and in particular, the Bilateral Symmetric Interference condition. This implies that the network's goals and frequency space form an exact potential game with a potential function which is a scalar multiple of the negation of the network interference function, $\Phi(f)$. The following provides detailed information about the proposed algorithm.

7.2.1 Algorithm Details

The proposed algorithm can be described as follows. Suppose each access node maintains a table with $|F_i|$ entries initialized to zero corresponding to the $|F_i|$ channels available to the network. Whenever access node i detects an RTS/CTS signal from another access node, j, i adds the power received from j to the table entry corresponding to the channel used by j. If j had been previously observed, then its previous received power is subtracted from its previous entry so that an access node only impacts a single table entry.

Now let us make the following assumptions about the network.

(A1) All RTS/CTS messages are transmitted at the same power level – a reasonable assumption as these messages are typically transmitted at maximum power to clear out hidden nodes.

(A2) The access nodes are not mobile so that $g_{ij} = g_{ji}$ or that power measurements are averaged over a long period so that $g_{ij} = g_{ji}$ is approximately true.

(A3) All channels have the same bandwidth.

(A4) A single access node adapts at a time.

(A1)-(A3) imply that $g_{ji} p_j s\left(f_i, f_j\right) = g_{ij} p_i s\left(f_j, f_i\right)$ which means that the bilateral symmetric interference of Chapter 6 holds. By the methods of Chapter 6, if each adaptation reduces the access node's observed interference, then the network is an IRN if one access node adapts at a time which is assured by (A4). While (A4) is a difficult condition to assure, if the frequency space is finite, then it can be shown that asynchronous adaptations converge to the same set of steady-states. As the network is an IRN, it is expected that this distributed algorithm will converge to a low interference steady-state which would be stable if the equilibria are isolated.

7.2.2 An 802.11h Application

As Chapter 6 asserts, since the only requirement on the decision process of cognitive radio is that adaptations increase (7.1) in order to decrease (7.2), great variation in the implementation of the decision process is permissible. Here we assume that each access node implements the following protocol:

1) Observe the spectral energy of the RTS/CTS messages of all observable access nodes.
2) At the time of its choosing, choose the channel on which the node observed the least amount of energy.

Consider a network of 802.11h access nodes (and presumably their client devices, but as the client devices are not involved in the decision process, they are irrelevant to the interactive decision problem). Suppose the access nodes are policy constrained to operate in the eleven channels available in the 5.47-5.725 GHz European band (channels 100-140) so that the assumption that "all RTS/CTS are transmitted at the same power level" holds for all channels (in this case, 1 W). Further, let us assume each radio has an equal probability of being the only radio allowed to adapt at each instance. As this is just a direct application of the general RTS/CTS DFS algorithm in Chapter 6 (where σ is now a binary function and discrete channels are used), we expect that the network will automatically sort itself into a low-interference frequency reuse pattern and that each adaptation will reduce the sum of perceived interferences in the network.

These expectations are confirmed in simulation of thirty access nodes randomly distributed over 1 km^2 operating in an environment with a path loss exponent of 3 with random placements and random initial channels and noise floors of -90 dBm. The geographic distribution of devices and their final operating frequencies are shown in Figure 7.2 where a circle notes the position of an access node with its final channel id labeled just below and to the right of the circle. Figure 7.3 depicts the operational channels for each access node (top), perceived interference levels by the access nodes (middle), and the sum of perceived interference levels (bottom) for the simulated network. Note that $\Phi(f)$ (bottom) decreases with each adaptation thereby satisfying the definition of an interference reducing network even though there are instances of interference increasing for individual access nodes (middle). Thus as is the case for all IRNs, self-interested adaptations led to a socially desirable outcome (at least when socially desirable is defined as the sum of observed network interference levels). As this algorithm converges to a minimum of $\Phi(f)$ (though not necessarily the global minimum), the algorithm performs at least as good as the centralized local search algorithm of [Leung_03] if no restarts are employed. So somewhat remarkably, this scalable distributed low complexity algorithm yields results as good as the high complexity centralized algorithm – a rare case of a “free lunch” in an engineering application.

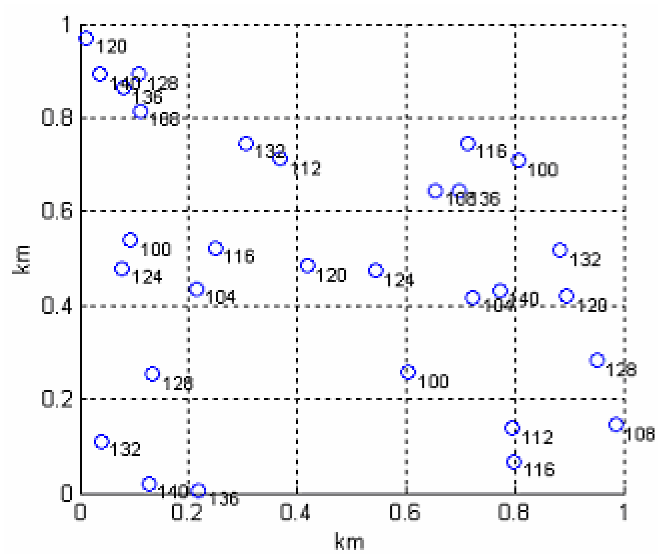

Figure 7.2: Steady-state Channels Selected for a Random Distribution of Access Nodes with Random Initial Channels in the 5.47-5.725 GHz Band.

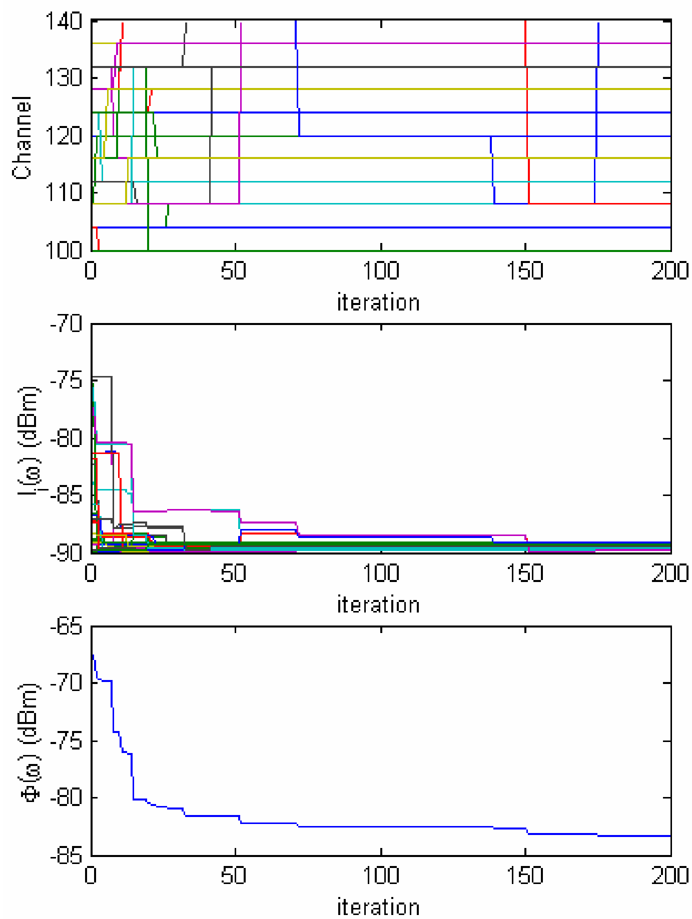

Figure 7.3: Instantaneous Statistics for Network in Figure 7.2.

7.3 Algorithm under Non-Ideal Conditions

In the preceding, we made a number of assumptions to make the network be an ideal IRN. As the following progressively relaxes these assumptions, it is seen that the proposed algorithm retains its desirable properties.

7.3.1 Policy Variations

If we permit the radios to choose permissible channels beyond channels 100-140, the assumption that all RTS-CTS messages are transmitted at the same power level fails as the lower and middle UNII bands (channels 36-64) limit transmission power levels to 200 mW [Etkin_05]. This violates (A1) ($p_k = p_i \; \forall i,k \in N$). However, for non-overlapping signals, $\sigma(f_i,f_k) = \sigma(f_k,f_i) = 0$, so BSI still holds and the network is still an IRN. Repeating the previous simulation and changing only the permissible channels and reflecting the transmission power policy variation we get the instantaneous statistics shown in Figure 7.4 where it is evident that the network continues to be an IRN.

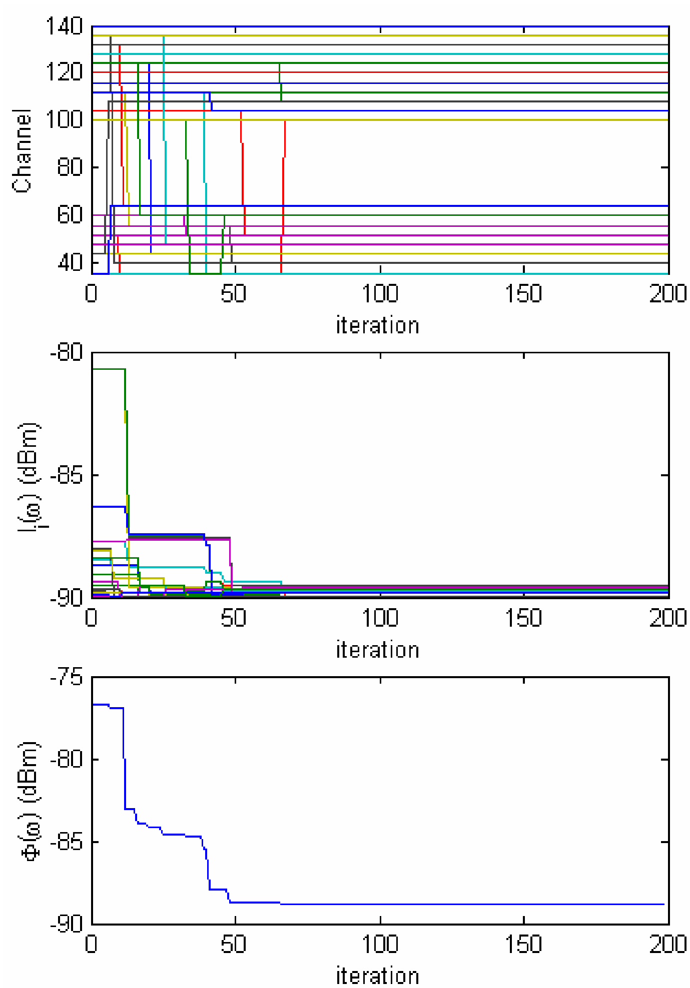

Figure 7.4: Instantaneous Statistics with Policy Variations

7.3.2 Asynchronous Timing

In the preceding, we assumed that one and only one access node adapted at any instance in time (A4). However, because adaptations and observation processes do not occur in infinitesimal periods of time it is likely that multiple access nodes will occasionally adapt simultaneously – a trend that becomes more likely as the number of access nodes in the network increase. So now continue to have the policy variations of the previous section and now assume that (A4) does not holds and instead of assume that each access has an opportunity to adapt at each iteration of the algorithm with non-zero probability.

Following the algorithm considered in this paper and the relaxed timing constraint two radios which are operating in the same channel and in close proximity to each other could simultaneously choose to adapt to another channel where a distant radio is operating. In this case, $\Phi(f)$ would increase even though each radio chose the channel which the radio had measured as having the least interference. Thus with (A4) relaxed, the proposed algorithm cannot be guaranteed to yield the strict monotonicity required by the definition of an IRN.

Yet this network will still converge to a steady-state with that is a minimizer of $\Phi(f)$. This again is a result of $<N,F,\{u_i\}>$ forming an finite exact potential game which implies FIP. As it is an exact potential game, minimizers of $\Phi(f)$ are Nash equilibria and the game has FIP which means that from any starting state, every sequence of self-interested unilateral adaptations must terminate in a minimizer of $\Phi(f)$. Due to these two properties, the network can be modeled as an absorbing Markov chain where minimizers of $\Phi(f)$ are the absorbing states of the chain. By virtue of being a minimizer, there can be no unilateral deviations that reduce interference; thus minimizers are absorbing states. By virtue of the finite improvement path property, there always exists a sequence of adaptations of non-zero probability that terminate in a minimizer as long as the probability of a unilateral deviation is always nonzero. Thus even with (A4) relaxed to asynchronous timings for adaptations, the network will still converge to a minimizer of $\Phi(f)$.

To verify this assertion, we modified the preceding simulation so that at each "iteration" each access node had an opportunity to adapt with probability 0.02. The instantaneous statistics for this simulation are shown in Figure 7.5. While $\Phi(f)$ still trends down, it longer does so monotonically. Nonetheless, because this system forms an absorbing Markov chain, it eventually converges to a frequency vector that is a minimizer of $\Phi(f)$.

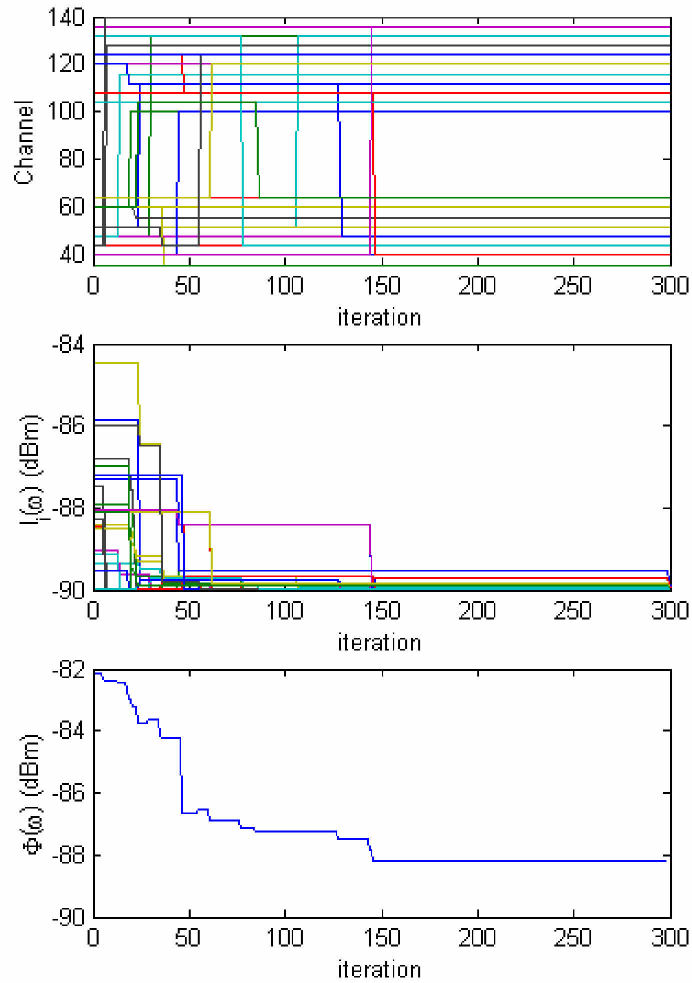

Figure 7.5: Impact of Asynchronous Decision Timings

7.3.3 Private Frequency Preferences

Throughout this discussion we have assumed via (A2) that each access node only intends to minimize the interference it perceives from other adaptive access nodes. However, because of the presence of interferers or because of local channel conditions, different access nodes may also exhibit different preferences for different frequencies. If we denote the frequency preferences of access node i as $S_i(f_i)$, these preferences might be incorporated as shown in (7.5).

$$\tilde{u}_i(f) = \sum_{k \in N \setminus i} g_{ki} p_k s(f_i, f_k) - S_i(f_i) \tag{7.5}$$

Note that $S_i(f_i)$ indicates that this component for access node i is only a function of access node i's choice of frequency and makes the most sense express additively as in (7.3) where $S_i(f_i)$ models the influence of static interferers.

Under the assumption that $S_i(f_i)$ models static interferers in the environment (7.2) no longer reflects the sum network interference. Instead sum network interference with frequency preferences is given by (7.6) .

$$\Phi^S(f) = \sum_{i \in N} \left(S_i(f_i) + \sum_{k \in N \setminus i} g_{ki} p_k s(f_k, f_i) \right) \tag{7.6}$$

This inclusion of additional interferers or jammers or local channel conditions may also impact bilateral symmetric interference as the interferers may not be transmitting at the same power level as the cognitive radios or may be operating with differing bandwidths.

Regardless of the loss of bilateral symmetric interference due to variances in the static interferers, $\langle N, \Omega, \{u_i\} \rangle$ remains an exact potential game but with an exact potential function given by (7.7).

$$V^S(w) = -\sum_{i=1}^{n} \left(S_i(f_i) + \sum_{k=i+1}^{n} g_{ki} p_k s(f_k, f_i) \right) \tag{7.7}$$

Note that the differences between (7.6) and (7.7) imply that the network is not strictly an IRN. Consider the scenario where a unilateral adaptation is made from a channel that is originally only occupied by the adapting access node i and a static interferer to a channel that is occupied only by access node k such that (7.8) holds.

$$g_{ki} p_k s(f_i, f_k) < S_i(f_i) < 2 g_{ki} p_k s(f_i, f_k) \tag{7.8}$$

This adaptation would increase (7.5) – thereby satisfying the proposed algorithm – but (7.6) would also increase – violating the definition of an IRN. However, the exact potential in (7.7) will always increase, ensuring the algorithm's convergence. And when the only maximizers of (7.7) are those for which $S_i(f_i)=0 \ \forall i \in N$, the algorithm will converge to a minimizer of (7.6) as under this condition $\Phi^S(f)= -2V(f)$. Even though it is trivial to construct two-access node, two channel, single interferer scenario with non-random geographic and channel distributions where (7.8) is satisfied, repeated trials of our randomly placed, random initial channel simulation have not yielded an adaptation that satisfies (7.8), which indicates the condition might be rare in practical settings. For example, modifying the policy variation simulation so it includes five static interferers operate in both channels 132 and 136, but distributed randomly geographically yield the simulation shown in Figure 7.6.

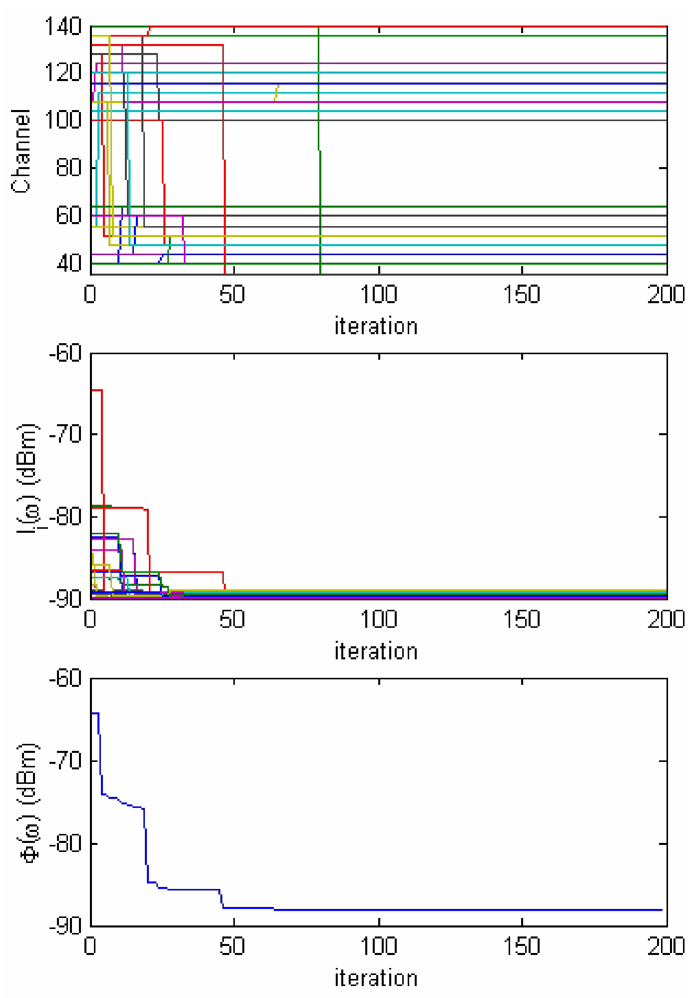

Figure 7.6: Algorithm with Private Frequency Preferences

7.3.4 Effect of Estimations

Throughout the preceding, we have implicitly assumed that the access nodes are perfectly measuring the signal strength of the RTS/CTS signals. However, in a practical setting, measurements of interference levels in differing channels would be corrupted by receiver noise, non RTS/CTS signals, and RTS/CTS signals too weak to decode and recognize as from an access node. Thus measurements of received power will at best be corrupted estimations. In such a scenario, the access nodes' goals would again take the form as shown in (7.5) but with $S_i(f_i)$ a stochastic variable.

As shown in the preceding section, a goal of the form of (7.5) implies that while $\langle N,F,\{u_i\}\rangle$ is still an exact potential game, the network will not necessarily remain an IRN for all possible realizations. Further, for channels with very low interference levels, $S_i(f_i)$ may be a dominant term and its natural time variation may spawn unnecessary adaptations.

For example consider a modification of the preceding simulation where the -90 dBm noise floor is implemented as a Gaussian stochastic variable. The results of this simulation are shown in Figure 7.7. While the algorithm still yields an almost 15 dB reduction in interference levels from the initial random distribution, $\Phi(f)$ is no longer monotonic, overall performance is decreased and significant bandwidth would be wasted signaling all of these adaptations. However, by modifying the algorithm so the access nodes only adapt if the improvement in performance is predicted to be more than a small threshold (-85 dBm), the system behaves as shown in Figure 7.8 – generally like a convergent IRN, but with the caveat that there exists the small probability that either an adaptation may increase sum interference. Based on the discussion of Chapter 5, we recognize this as an ε-better response decision.

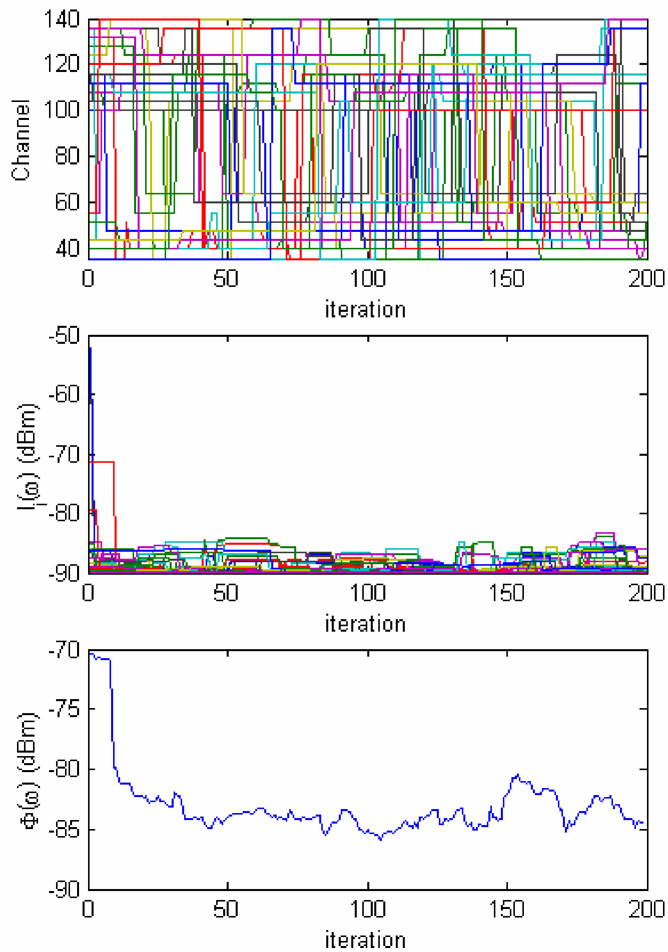

Figure 7.7: Algorithm with Stochastic Estimations.

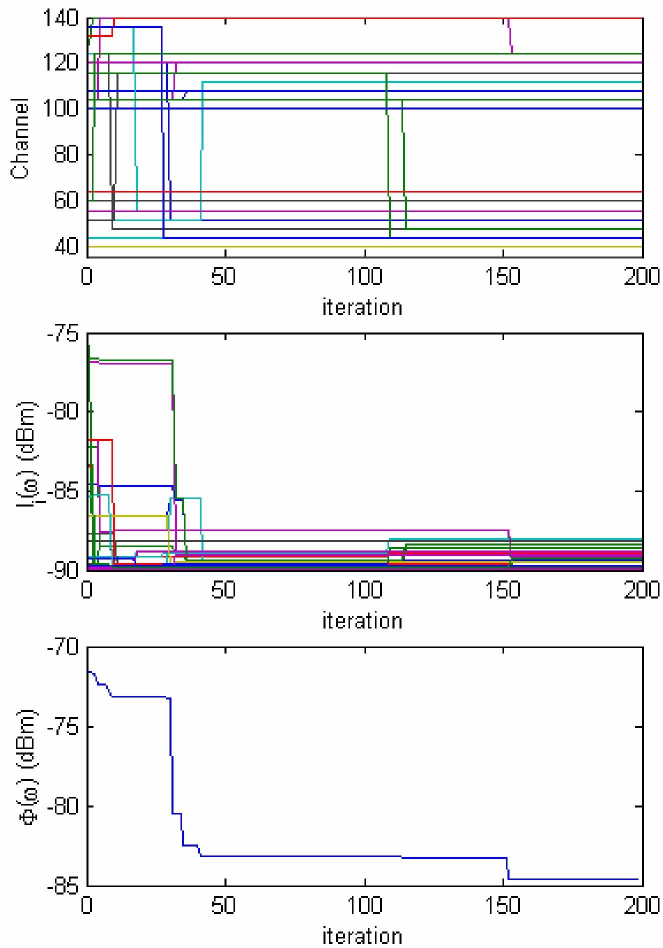

Figure 7.8: Algorithm with Stochastic Estimations and a small adaptation threshold (-85 dBm)

While this modified decision rule is stable, it is somewhat at odds with how we have been treating the stability of decision rules. Specifically, we have been primarily operating under the assumption that each decision rule can be characterized as $d_i : A \to A_i$ and characterizing stability as for every ε>0 there exists some δ>0 such that if the system is perturbed off the steady state, a^* by a distance no greater than $\|a^* - a\| < d$ then the system will remain no further away from a^* than $\|a^* - a\| < e$ absence further perturbances. Because of the finiteness of the action space and the lack of isolated steady-states, this formulation is impossible.

Yet the system is stable as shown in the simulation. Recall that in the model in Chapter 2 we said that expressing the decision rule as a function of the action space was really only a useful analytic conceit for decision rules that are a function of the outcome space and that the radios were really observing and reacting to the outcome space. So rather than $d_i : A \rightarrow A_i$ the radios are actually implementing $d_i : O \rightarrow A_i$. When we introduce noise to the observations of the radio, this has the effect of perturbing the observed outcomes in *O*. So we are actually referring to the stability of $d_i : O \rightarrow A_i$. With this in mind, it is relatively easy to show that the threshold causes d_i to be stable. Because of the threshold in the decision rule and assuming the system is at an NE, it now takes a perturbation in the outcome space at least as great as the threshold to induce an adaptation. Or more formally, for any arbitrarily small $\varepsilon>0$ and assuming that the system is operating at an NE, for all $\tau>\delta>0$ the system remains within an ε of the steady-state, specifically, the system remains on the original steady-state. Note that the thresholded decision rule induces many more equilibria in the system (specifically a number of ε-NE) so different steady-states may have smaller permissible values for δ.

7.3.5 TPC and DFS

[Etkin_05] states that TPC is intended to support variations in policy and adaptations based on "a range of information including path loss and link margin." As we showed in the Policy Variations section, as long as it is applied consistently across a channel policy variations do not impact the IRN features of the algorithm. However, if the RTS/CTS power levels are set at varying levels by the differing access nodes operating in the same channel, then it is likely that (A1) will be violated in situations where $s(f_i, f_k) \neq 0$ which means the BSI condition will not be satisfied. For instance, consider a modification of the original policy variation simulation where the transmit power that each access node applies to its RTS/CTS signals has been scaled by a factor randomly drawn from a clipped Gaussian distribution (clipped so as to rule out negative power levels) whose results are shown in Figure 7.9. Note that $\Phi(f)$ does not decrease monotonically in this simulation, though it does trend downwards fairly consistently and converges for all simulations to date. When TPC is applied to the RTS/CTS messages, it is observed that the system still converges to an interference minimizer. Currently, we do not have a firm analytic explanation for this phenomenon, though it is known that for relatively small variations in transmit power levels, $\Phi(f)$ will be an ordinal potential function for $<N, F, \{u_i\}>$ so for many realizations of TPC applied to RTS/CTS signals the network will still behave as an IRN. However, without a firm analytical basis for stating why desirable behavior results we are unable to rule out unforeseen pitfalls from the interactions. So we recommend that application of the proposed algorithm be limited to scenarios where TPC is applied only to the DATA and ACK messages. While this assumption would still enable improved battery life and would be consistent with the RTS/CTS messages original intent for clearing out hidden nodes, it would limit the gains seen from frequency reuse. However, all localized TPC schemes face a functionality tradeoff of clearing out hidden nodes versus maximizing frequency reuse. By reducing transmit power on the RTS/CTS message, a higher cluster density can be achieved, but this comes at a cost of increasing the probability that a hidden node will miss the RTS/CTS signal and subsequently interfere with the data transfer, particularly where TPC is guided by local decisions instead of policy.

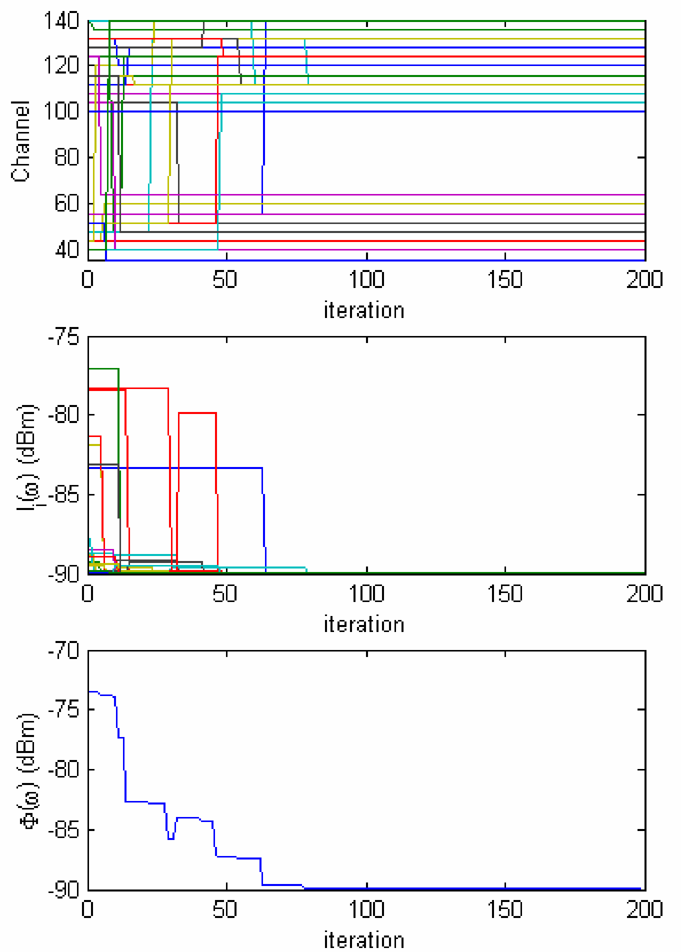

Figure 7.9: Algorithm with TPC Applied to RTS/CTS.

7.4 Summary and Conclusions

This chapter presented a novel algorithm for performing DFS which does not require the use of a centralized controller, specialized network topologies, or even any message passing between nodes but still achieves performance as good as could be expected from a centralized heuristic algorithm, e.g., [Leung_03]. The remainder of this section summarizes the results of the DFS algorithm, describes how game theory aided the design, and describes further extensions that can be made to this algorithm.

7.4.1 Algorithm Summary

By leveraging the framework of interference reducing networks, this chapter proposed a low complexity autonomous distributed ad-hoc DFS algorithm whose adaptations converge to a minimizer of the sum of observed interference levels by minimizing their own perceived interference measured from the RTS/CTS signals of other access nodes. We showed that this non-cooperative non-collaborative algorithm is robust to policy variations, timing variations, the presence of interferers, and noisy estimations of signal strengths when a simple adaptation threshold is applied to the algorithm. Though empirically convergent, when TPC is applied to the RTS/CTS signals, the algorithm fails to satisfy the IRN framework. However, the assumption of TPC applied to RTS/CTS signals may not be realistic as it necessarily increases susceptibility to hidden nodes. While all simulations implemented a best-response dynamic, any self-interested decision rule – including an ontological reasoning engine – will converge by virtue of being an exact potential game and an IRN.

7.4.2 How Game Theory Aided the Design

This chapter demonstrated what can be gained by leveraging the techniques of the previous chapters. We knew that if a cognitive radio network could be designed as a potential game, then unilateral deviations would converge and the maximizers of the system's potential function would be steady-states. The one hole in applying potential games is that convergence is not guaranteed to be to a desirable steady-state. As noted in Chapter 5, this problem can be solved by designing networks where the potential function is the design objective function. Chapter 6 gave the framework for doing this wherein the concept of bilateral symmetric interference guaranteed that self-interested interference minimizing adaptations yield an interference reducing network. The condition of bilateral symmetric interference was an application of bilateral symmetric interaction exact potential game to cognitive radio networks. Asynchronous convergence was then assured by the FIP convergence analysis of Chapter 4 which was in turn aided by the Markov models of Chapter 3. We also knew that the network would be stable – a concept introduced in Chapter 3 – because of the consideration of the stability of ε-better responses in finite potential games.

Because of game theory and the results established in the preceding chapters, we knew that a low complexity, scalable DFS algorithm, convergent, stable, and desirable network would result if we could get the network to satisfy the bilateral symmetric interference condition. And because of game theory and the earlier results we knew that this would occur without having to resort to a centralized controller, without message passing between the radios so they could all independently find the same solution, and without resorting to specialized network topologies.

With all this in hand, the only insight needed to design this network was identifying a situation where symmetric link gains and equal transmit powers could reasonably be assumed to be present. And this is satisfied simply by having each access point tabulate the channel and received power of the RTS/CTS messages of other access points which it can observe.

Beyond the single best response decision rule presented in the original paper and covered in this chapter, the FIP results of Chapter 4 and Chapter 5 also assure us that the combination of decision rules and timings whose entries contain a 'Y' in Table 7.1 also converge. Thus with the correct observations, goals, and action space in place many different scenarios are known to converge.

Table 7.1: Other Conditions Guaranteed to Converge to a Low Interference State

	Timings			
Decision Rules	Round-Robin	Random	Synchronous	Asynchronous
Best Response	Y	Y	N	Y
Exhaustive Better Response	Y	Y	N	Y
Random Better Response[(a)]	Y	Y	Y	Y
Random Better Response[(b)]	Y	Y	N	Y
ε-Better Response[(c)]	Y	Y	N	Y
Intelligently Random Better Response	Y	Y	N	Y

(a) Proposed random better response (b) Random better response of [Friedman_01] (c) Convergence to an ε-NE

7.4.3 Further Extensions

This algorithm need not be specifically limited to finite channel sets (though the table entry routine would require modification) nor to nonoverlapping channels as relaxing these assumptions still preserves the bilateral symmetry assumption that $s(f_i, f_k) = s(f_k, f_i) = \max\{B - |f_i - f_k|, 0\}/B$. For instance, the same algorithm could be readily applied to a 2.4 GHz 802.11b network which has 11 channels where at most 3 channels (1, 6, and 11) can be made to not overlap. However, because the bilateral symmetry assumption still holds, the observation of access nodes' RTS/CTS signals will still satisfy bilateral symmetric interference.

Obviously, this algorithm can be extended to the other decision rules and timings listed in Table 7.1. Because the random better response decision rules converge, such an algorithm could be easily inserted into genetic algorithm based cognitive radios and used as part of a rapidly deployed network of radios under the assumption that symmetric link gains and constant observed power levels still hold. Further, because all exhaustive better response algorithms converge, it seems reasonable that if the key observation (RTS/CTS signals of access nodes) is used to drive the decision process of ontologically defined cognitive radios, an interference reducing network will still emerge.

So in the end, we can replace the need for a wireless PhD student on every street with the simple requirements that 1) each access point notes the received power and channels of the RTS/CTS messages from other access points that it can observe and 2) each access point acts to reduce its own perceived interference.

7.5 References

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Chapter 8: Applications of Weak FIP

"Two roads diverged in a wood, and I,
I took the one less traveled by,
And that has made all the difference."
- R. Frost, The Road Not Taken

In a cognitive radio network, each cognitive radio repeatedly makes choices that impact the evolution of the network state. But unlike Robert Frost, cognitive radios may have the chance to make the same choice again and again if play does not converge. However, in networks that can be modeled as a game with weak FIP, there always exists at least one choice that "makes all the difference" and adaptations will lead to an NE.

In Chapter 4, we identified weak FIP as a critical property for the convergence of cognitive radio networks. Specifically, we asserted that without weak FIP, a cognitive radio network could not be guaranteed to converge to an NE if the radios are making myopic individually rational adaptations under round-robin or random timing. However, for games with weak FIP there always exists some improvement path that leads to an NE, so this repetition must come to an end in a game with weak FIP.

While we have made an extensive discussion of FIP (which implies weak FIP), we have not discussed any specific techniques for identifying when a network has weak FIP without FIP nor have we presented any cognitive radio applications of weak FIP. This chapter addresses these shortcomings and discusses a readily identified game model that can be used to establish that a game has weak FIP (Section 8.1 – Supermodular games), and two cognitive radio applications where weak FIP occurs – ad-hoc power control (Section 8.2) and sensor network formation (Section 8.3). For practical considerations, this chapter does not provide the same deliberate presentation given in the preceding chapters. However, the most widely applicable insights into how to identify when a cognitive radio network has weak FIP are covered – demonstration that the game is a finite supermodular game and identification of an everywhere convergent improvement path..

8.1 Supermodular Games

A normal form game, $\Gamma = \langle N, A, \{u_i\} \rangle$ is termed a *supermodular game* if the action space forms a *lattice* and the utility functions are *supermodular*. A partially ordered set, X, is termed a lattice if for all $a,b \in X$, $a \wedge b \in X$ and $a \vee b \in X$ where $a \vee b = \sup\{a,b\}$ and $a \wedge b = \inf\{a,b\}$. A function $f : X \to \mathbb{R}$ where X is a lattice, is termed supermodular if for all $a,b \in X$, $f(a) + f(b) \leq f(a \wedge b) + f(a \vee b)$.

8.1.1 Model Identification

While the definition may seem complicated, a game can be identified as a supermodular game if all players' utility functions satisfy the relationship given in (8.1) and the action space is compact subset of real space [Milgrom_90]. Such a game is called a *smooth supermodular game.*

$$\frac{\partial^2 u_i(a)}{\partial a_i \partial a_j} \geq 0 \forall j \neq i \in N \tag{8.1}$$

8.1.2 Steady-states

As shown in [Topkis_98], the best response function for a supermodular game is a monotonic function of a, i.e., $a^1 \geq a^2 \Rightarrow \hat{B}(a^1) \geq \hat{B}(a^2)$. By Tarski's fixed point theorem given in [Topkis_98], monotonic functions on a compact space (not necessarily convex) have a fixed point. Thus the best response function for a supermodular game has a fixed point, which implies the game must have at least one NE.

By [Topkis_98], all NE for a game form a lattice. While this does not particularly aid in the process of initially identifying NE, from every pair of identified NE, e.g., a^* and b^*, additional NE can be found by evaluating $a^* \wedge b^*$ and $a^* \vee b^*$. In general, NE identification for a supermodular game has to proceed as it did in Chapter 4 – by simultaneously solving the system of best response equations for fixed points.

More usefully, it is possible to establish a condition where a supermodular game has a unique NE by leveraging the Standard Interference Function of [Yates_95] which is an example of a monotonic best response function. This novel result turns out to be very useful for convergence and stability properties of supermodular games.

Theorem 8.1: NE Uniqueness in a Supermodular Game (*)

Given a supermodular game with a real convex compact action space, suppose the best response function satisfies the following conditions:

1. *Uniqueness* - $\{b_i \in A_i : u_i(b_i, a_{-i}) \geq u_i(a_i, a_{-i})\ \forall a_i \in A_i\}$ is a singleton for all a.
2. *Positivity* - $\hat{B}(a) > 0$
3. *Scalability* – For all $\alpha > 1$, $\alpha \hat{B}(a) > \hat{B}(\alpha a)$.

then the game has a unique NE.

Proof: (Paralleling the proof in [Yates_95] that the Standard Interference Function has a unique fixed point) By Tarski's fixed point theorem, an NE exists for this game. Now suppose that a^* and b^* are both fixed points. By positivity, we know that $a_i^* > 0$ and $b_i^* > 0$ for all $i \in N$. As these are distinct fixed points, there must be some $a_i^* > b_i^*$ (or some $b_i^* > a_i^*$, simply interchange the following comparisons). By scalability, there exists $\alpha > 1$ such that a) $\alpha a^* \geq \alpha b^*$ and b) for some i $b_i^* = \alpha a_i^*$. Then by the monotonicity of $\hat{B}$ and the scalability property, it must hold that $b_i^* = \hat{B}_i(b^*) \leq \hat{B}_i(\alpha a^*) < \alpha \hat{B}_i(a^*) = \alpha a_i^*$, a contradiction as $b_i^* = \alpha a_i^*$.

8.1.3 Desirability

In general, little can be said about the desirability or optimality of a supermodular game's NE. However, as we saw for potential games in Chapter 5, an NE in a supermodular game whose utility function satisfies (8.1) can be adjusted by introducing any additive self-motivated function.

8.1.4 Convergence

By [Friedman_01], finite supermodular games have weak FIP, i.e., from any initial action vector, there exists a sequence of selfish adaptations that lead to a NE. Thus the convergence results of Chapter 4 for games with weak FIP apply to supermodular games.

A variation on the simultaneous best response algorithm is presented in [Milgrom_90] wherein the players follow what is termed an *adaptive dynamic process*. In an adaptive dynamic process, all players play a best response to some arbitrary weighting of recent past actions by other players.

Definition 8.1: *Adaptive dynamic process* ([Milgrom_90] (A6))
Formally, a decision rule is defined as an *adaptive dynamic process* if $\forall t^* \in T$ there exists a $t' \in T$ such that $\forall t \geq t'$, $d_i\left(a^t\right) \in \bar{U}\left(\left[\inf\left(P\left(t^*,t\right)\right),\sup\left(P\left(t^*,t\right)\right)\right]\right)$ where $P(t^*, t)$ denotes the action tuples observed between times t^* and t, $U(a)$ is the list of undominated responses to a for each player, and $\bar{U}(a) = \left[\inf\left(U(a)\right),\sup\left(U(a)\right)\right]$.

The corollaries to Theorem 8 in [Milgrom_90] show that a smooth supermodular game following an adaptive dynamic process with any timing converges to a region bounded by the Nash equilibrium lattice and that iterative elimination of dominated strategies converges to a region defined by the Nash equilibrium lattice. Note that when the NE is unique, the adaptive dynamic process converges to the NE.

8.1.5 Stability (*)

In general little can be said about the stability of a supermodular game. However, if the best response function satisfies Theorem 8.1, then the unique NE will be asymptotically stable under best response decision rules with any timing with a Lyapunov function given by $L(a) = \left\|a^* - a\right\|$ - the same Lyapunov function we defined for the standard interference function. This can be quickly verified by noting that the best response decision rule is monotonic under any timing [Topkis_98] and the uniqueness of the NE implies that every adaptation must bring the network closer to the NE. Interestingly, this implies that the best response function decision rule in a supermodular game constitutes a *pseudo-contraction* – a topic covered in Chapter 3.

8.2 Ad-hoc Power Control[1]

As we showed in Chapter 5, when cognitive radios distributed implement power control at a single cluster head where each radio is guided by the utility function given in (8.2), the system forms an ordinal potential game.

$$u_i(\mathbf{p}) = -\left|\hat{g} - \frac{g_i p_i}{1/K\left(\sum_{k\in N\setminus i} g_k p_k + s\right)}\right| \quad (8.2)$$

Unfortunately, this result is not so easily extended to ad-hoc networks. Yet, we can still establish broad convergence and stability results for target-SINR power control algorithms in ad-hoc networks.

[Neel_05], considers the analysis of distributed power control in an ad hoc network where each link, j, varies its transmit power in an attempt to achieve a target SINR, γ_j, measured in dB. This scenario can be thought of as analogous to the fixed assignment scenario presented in [Yates_95]. Indeed this analysis can be considered an extension of [Yates_95] to ad-hoc networks with additional consideration given to stability.

8.2.1 Stage Game Model

Based on the preceding discussion, a normal form stage game can be formulated as follows.

- Player Set N – Set of decision making links
- Player Action Set A_j – The real convex, compact set of powers, $\left[0, p_j^{\max}\right]$ where $p_j^{\max}$ is the maximum transmit power of cognitive radio j. The action space, $A \subset \mathbb{R}^n$, is given by $A = A_1 \times A_2 \times \cdots \times A_n$.
- Utility – An appropriate action based utility function for a target SINR (dB) algorithm is given by (8.3) where $\hat{g}_j$ is the SINR target of cognitive radio j.

[1] This text is taken from [Neel_06].

$$u_j(\mathbf{p}) = -\left(\hat{g}_j - 10\log_{10}(g_{jj}p_j) + 10\log_{10}\left(\frac{\sum_{k\in N\setminus j} g_{kj}p_k + s_j}{K}\right)\right)^2 \qquad (8.3)$$

Here, communications theory provides the necessary connection between action and outcome as SINR. Using the notation we presented in Chapters 3 and 5, in a network, *N*, of cognitive radios the SINR of the signal transmitted by *j* and received by its node (radio) of interest measured in dB is given by (8.4) where g_{kj} is the effective fraction of power transmitted by node *k* that is received at *j*'s node of interest (receiving end of *j*'s link) and N_j is the noise at the receiving end of link *j*.

$$g_j = 10\log_{10}(g_{jj}p_j) - 10\log_{10}\left(\sum_{k\in N\setminus j} g_{kj}p_k + N_j\right) \text{(dB)} \qquad (8.4)$$

8.2.2 Analysis

In [Altman_03] it is claimed that the cellular fixed assignment scenario of [Yates_95] on which this ad-hoc network model is based is supermodular. The following parallels the analysis in [Neel_05] where we showed that this stage game constitutes a supermodular game for an ad-hoc network.

A stage game can be shown to be a smooth supermodular game by applying the second order conditions we presented in Section 8.1. First, notice that the action space forms a complete lattice (compact subset of Euclidean space). Then evaluating the second derivative with respect to p_j and p_k where *k* is any cognitive radio $k \in N \setminus j$ yields (8.5).[2]

$$\frac{\partial^2 u_j(p)}{\partial p_j \partial p_k} = \frac{200 g_{kj}}{p_j\left(\sum_{k\in N\setminus j} g_{kj}p_k + N_j\right)\ln(20)} \qquad (8.5)$$

As (8.5) is strictly positive, the last conditions for a smooth supermodular game is satisfied. Accordingly, we know that the network

- Has at least one steady state and

[2] Note that as g_{kj} will not generally equal g_{jk}, this game will not be an exact potential game.

- Converges for synchronous and asynchronous best response algorithms (local optimization).

As (8.5) is not a function of the target SINRs, each radio can have its own target SINR and the game will remain a supermodular game.Further, since the best response algorithm given by (8.6) is a known standard interference function (or equivalently since $\hat{B}$ satisfies Theorem 8.1), we know the following:

- The network has a unique fixed point.
- The network achieves the target SINR vector with the smallest possible power vector (when the SINR vector is feasible).
- A Lyapunov function is given by the distance between the current power vector and the fixed point.

$$\hat{B}_j(\mathbf{p}) = p_j \frac{\hat{g}_j}{g_j} \quad p_j^{t_{k+1}} = p_j^{t_k} \frac{\bar{g}_j}{g_j} \qquad (8.6)$$

Finally, for a feasible SINR target vector, the unique steady state for this game can be found by solving the linear system of equations $\mathbf{Z}\vec{\mathbf{p}} = \mathbf{?}$

where $\mathbf{Z} = \begin{bmatrix} h_{1n_1} & -\hat{g}_1 h_{1n_2} & \cdots & -\hat{g}_1 h_{1n_n} \\ -\hat{g}_2 h_{2n_1} & h_{2n_2} & \cdots & -\hat{g}_2 h_{2n_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{g}_n h_{nn_1} & -\hat{g}_n h_{nn_2} & \cdots & h_{nn_n} \end{bmatrix}$, $\vec{\mathbf{?}} = [\hat{g}_1 N_1 \quad \hat{g}_2 N_n \quad \ldots \quad \hat{g}_n N_n]^T$, and $\vec{\mathbf{p}} = [p_1 \quad p_2 \quad \ldots \quad p_n]^T$. Here, $h_{jk} = g_{jk}/K$ for $j \neq k$ and $h_{jj} = g_{jj}$.

To determine the desirability of this fixed point, we can consider a network design objective function that seeks the minimum power vector that provides at least the target SINR for all links. For this design objective function, the fixed point power vector is optimal as no smaller power vector achieves the target SINR (assuming the target SINRs are feasible) because otherwise the smaller power vector would also be a fixed point in violation of a property of being a Standard Interference Function.

8.2.3 Validation

Consider the ad hoc network shown in Figure 8.1 where at a particular frequency each terminal is attempting to maintain a target SINR at a cluster head and each cluster head is maintaining a target SINR at the gateway node. The signals employed by the radios have a statistical spreading factor of *K*.

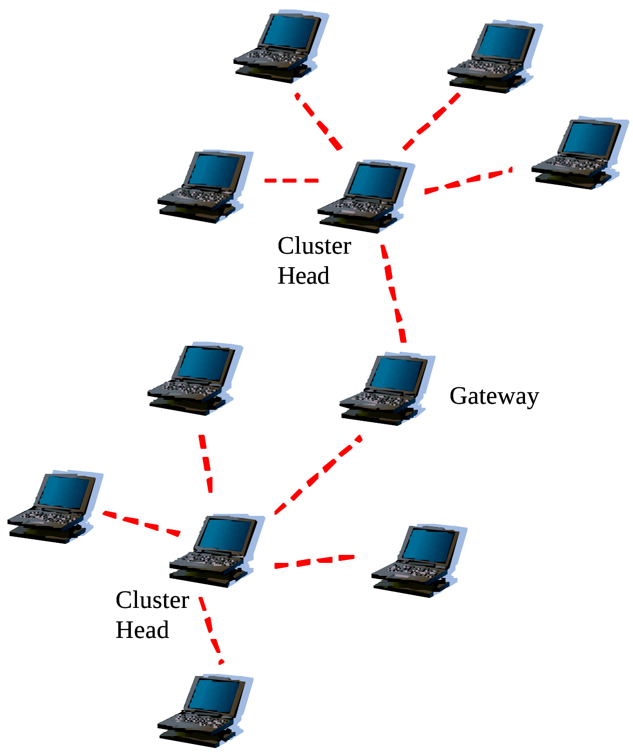

Figure 8.1: Simulation scenario for ad-hoc power control example

Assuming these devices are adjusting their power levels in a locally optimal manner, then the network conforms to the model described in the preceding. Accordingly, we would expect that any initial power vector would converge to a unique power vector and that even when corrupted by noise, the system would remain in a region near this steady state as the network is Lyapunov stable.

A simulation was constructed for deterministic and stochastic simulation scenarios. The simulation results for these scenarios are shown in Figure 8.2 and Figure 8.3, respectively. Note that the locally optimal algorithm rapidly converges to the steady state in both scenarios and that even in the presence of random noise-induced perturbations, the network remains in a region around the deterministic steady-state. However, as in

Chapter 7, adaptations could be further smoothed by introducing a threshold to the decision rule. Further, rather than the best response decision rule employed in this simulation, the radios could also be implementing an averaged best response (a particular example of an adaptive dynamics process).

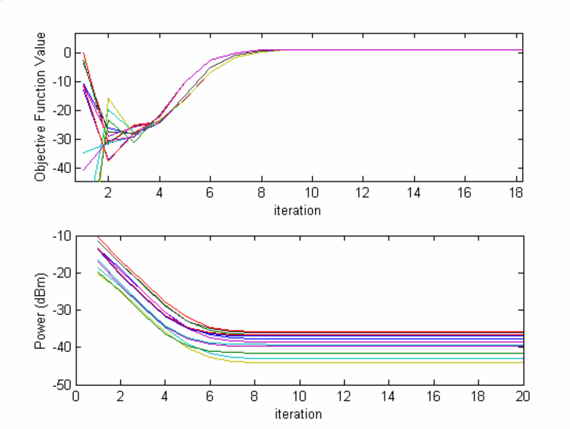

Figure 8.2: Noiseless simulation.

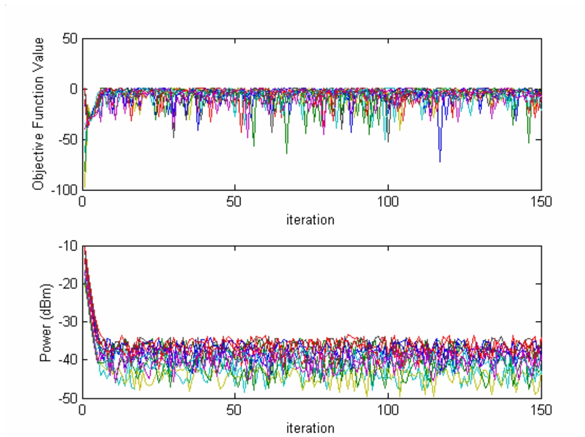

Figure 8.3: Noisy simulation.

8.3 Sensor Network Formation (*)

Consider a network of sensors collecting information and charged with transporting the information back to some data base (the information *sink*). It is frequently more efficient (in terms of cost, battery life, coverage, and covertness) for the radios to indirectly transport their information to the sink. In this example, we study a collection of wireless sensors which are guided by a desire to transport their data back to a common data sink (perhaps connected to the IP cloud) balanced against a desire to minimize power consumption – a term which is assumed to be dominated by transmit power. Because the sensors contain recording devices, if the cost to transmit is too high, the sensors have the option of simply storing the data for later retrieval.

Under the assumptions that forwarding other sensors' data is costless and that only a single sink is present, the following presents a model of this situation and a proof that the network has weak FIP.

8.3.1 Model

We can form a game model for this network as follows. We consider the player set to be the set of sensors (or nodes), N, and assume that each sensor i can form any number of directional links to the other sensors $j \in Ni$ (perhaps via beam forming) and denote a particular link between from i to j by the symbol l_{ij}. Each sensor has an action set, L_i, given by the power set of $\{l_{ik}\} \forall k \in N \setminus i$ (i.e., a sensor can choose to maintain any combination of links to other sensors or no link at all.) Each sensor network expresses its utility as a function of the network g which is defined by the set of links implemented by each player. In general $g \subset g^N$ where, $g^N = \{l_{ij} \mid \forall i, j \in N, i \neq j\}$ is the complete network. For notational convenience, we also define g_i as the set of directed links in g formed by node i, i.e., $g_i = \{l_{ij} \in g\}$, and g_{-i} as the set of all links in g other than g_i, i.e., $g_{-i} = g \setminus g_i$.

Let l^k denote a particular link and for l_{ij} we refer to i as the *originating node* and j as the *terminating node*. We say that $g = (l^1, l^2, \ldots, l^m)$ is a *path* in the network g if the originating vertex for every l^{k+1} is the terminating node for every l^k. We say that i is *connected* to j if there exists a path from i to j and refer to this path as ij. If i is connected to j via a path consisting of a single link, then we say that i is *directly connected* to j and refer to the single link as a *direct connection*. If i is connected to j yet there exists no direct connection between i and j, then we say that i is *indirectly connected* to j. We say that g is a *connected network* if every $i \in N$ is connected to every $j \in N$.

One particularly useful utility function model for sensor network formation is the link connections model. Originally, introduced in [Jackson_96], a slightly modified network valuation function for the link connections model is given by (8.6)

$$u_i(g) = \sum_{j \in N \setminus i} b_{ij} d_{ij}^{t_{ij}} - \sum_{l_{ij} \in g_i} c(l_{ij}) \qquad (8.6)$$

where b_{ij} is the benefit i receives for being connected to j, d_{ij} is a hop decay factor (perhaps reflecting increasing probability of transmission failure as the length of a connection increases from queue overflow or link failures) between i and j, t_{ij} is the length of shortest path between i and j, with $t_{ij} = \infty$ when no path exists, and $c(l_{ij})$ is the cost to sensor i of forming link l_{ij}.

This cost parameter could be simplified as $c(l_{ij}) = c_i$ where each player has a cost that is applied across all local links. This situation is encountered when cost is a function of transmit power and the nodes can only select a fixed transmit power for all links. Another appropriate choice of cost parameter is $c(l_{ij}) = a_i d(l_{ij})^n$ where $d(l_{ij})$ is the Euclidean distance between i and j, n is the path loss exponent, and a_i is the free space loss factor. This can be used to model situations where a particular received power must be achieved at the terminating end of a link and the environment has a uniform path loss model. In general, however, $c(l_{ij})$ will vary by link in a sensor network due to various obstructions that may make the signal propagation environment non-uniform.

For a sensor network we assume there exists some $k \in N$ (the sink) such that for all $i \in N \setminus k$ $b_{ik} = b$ and $b_{ij} = 0 \, \forall j \in N \setminus k$, i.e., there is one and only sensor to which every other sensor assigns a benefit of being connected. For this example we assume δ=1 so no information degradation occurs over multiple hops - a reasonable assumption with sufficiently large queues and sufficiently long periods of time for retransmissions.

8.3.2 Steady-states

The study of network formation in game theory (primarily social network) is somewhat unique in the sense that numerous network stability concepts have been introduced. These include *Nash networks*, *pairwise stability*, *link deletion proofness*, and *link addition proofness*. For this example where we are assuming that each sensor can arbitrarily change its links with each adaptation, the concept of the *Nash network* is the most natural equilibrium concept.

Definition 8.2: *Nash Network*
A network, g, is a *Nash network* if for every node $i \in N$, $u_i(g_i, g_{-i}) \geq u_i(g'_i, g_{-i})$ for all $g'_i \in L_i$.

In other words, a Nash network is a network where no node can improve its payoff by unilaterally altering any combination of its links.

Determining if and when a Nash network exists for this model of sensor network formation can be best done by simultaneously establishing a convergence condition performed in the following section. However, we can make some characterizations about the topology of any Nash network because we know that each sensor's best response is either a single link or no link (the utility function expresses no benefit for maintaining a redundant path). Thus in a Nash network, no paths branch (as that requires two outbound links) and no paths lead out from sink (as that would be costly to the sink without any benefit). Thus for this model of sensor network formation, all Nash networks lack cycles – paths such that some sensor *i* is indirectly connected to itself.

8.3.3 Convergence

To establish convergence of this sensor network to a Nash network we demonstrate that when the radios implement best response decision rules under round-robin timing, the network must progress through a sequence of readily characterized networks concluding in a Nash network. The first network we consider in this sequence is a trimmed network.

Definition 8.3: *Trimmed network* (*)
A network, *g*, is said to be a trimmed network if all sensors maintain no more than one link.

Now consider any initial distribution of links and what happens after every sensor has had a chance to implement its best response.

Theorem 8.2: Convergence to a Trimmed Network (*)
After a complete round-robin best response, any starting network must be in a trimmed network.
Proof: As there is no benefit to maintaining multiple links, the best response for every sensor is a single link or no link. So after each node has had a chance to play its best response, each node must have a single link or no link.

To further the convergence analysis, we must establish that certain paths are not contained in the network.

Definition 8.4:*Poison path* (*)
A path is said to be a *poison path* if it is a profitable path for some sensor *j* but is not profitable for some sensor after *j* in the path.

With a poison path in the network, the network cannot be a Nash network as at least one sensor will have to adapt. Further, other sensors may make adaptations which will have to be later reversed because of the unstable path to the sink. Fortunately, poison paths quickly disappear from any sensor network under round-robin best responses.

Theorem 8.3: Withering of Poison Paths (*)
For δ=1, trimmed sensor networks playing a best response do not create new poison paths.
Proof: With a trimmed sensor network and δ=1, creating a new poison path implies that a path exists to the sink such that some sensor has added a link where the cost outweighs the benefit. However, not playing any link is strictly preferable to creating such a link so creating a poison path cannot be a best response so a new poison path cannot be created.

With no poison paths being created, it is valuable to consider the situation where all poison paths are eliminated the network, a condition we term a *pruned network*.

Definition 8.5:*Pruned network* (*)
A network *g* is a *pruned network* if it contains no poison paths.

Starting from an arbitrary network, we can show that the sensor network rapidly converges to a pruned network.

Theorem 8.4: Convergence to a Pruned Network (*)
Starting from any initial network, the finite sensor network with δ=1 is pruned after the first best-response round-robin and remains pruned thereafter.
Proof: After the first round, the network is trimmed and no node maintains a link whose cost outweighs the potential benefit of having a path to the sink (if it did, then it would not be preferable to not playing a link implying the network is not trimmed). As this is a requirement for a poison path in a sensor network, no poison paths can exist and the network is pruned.

While there may not be any poison paths after the first complete round-robin, there may be sensors which do not have a path to the sink either because the sensor has no links or that one of its paths was broken when an unprofitable link was eliminated. Some portions of the network, however, may have positive utility.

Definition 8.6:*Healthy network* (*)
A *healthy network* is a pruned network in which every sensor has positive utility

In general, the sensors in a healthy network constitute a subset of N and the number of sensors in a healthy network is a nondecreasing sequence for round-robin best responses. For this sensor network, membership in the healthy network implies a path to the sink as such a path is required for positive utility.

Theorem 8.5: Healthy Network Stability (*)
For δ=1, all sensors in a healthy network remain in a healthy network when playing a round-robin best response (or better response) to a pruned network.
Proof: A sensor falls out of a healthy network if it chooses to disconnect or a sensor ahead of it in its path to the sink disconnects. As disconnecting drops the sensor's utility to 0 (or worse), this is never preferable as membership in the healthy network implies positive utility. Further no downstream sensor will profitably disconnect as this implies the existence of a poison path which contradicts the assumption of a pruned network.

We can make an interesting characterization of the sensor network after one round of best responses to the pruned network.

Theorem 8.6: Sensor Network Link Characterization (*)
After the first round of best responses to a pruned network, each sensor in the network is either in a healthy network or has no links.
Proof: By Theorem 8.5, once a sensor is part of a healthy network, its best response is to remain part of the healthy network. If it is not part of the healthy network and joining the healthy network would yield a positive utility, then the best response is to join the network. If a sensor cannot profitably join the healthy network, then utility is maximized when no link is implemented.

For those sensors in the healthy network, we can establish a convergence condition.

Theorem 8.7: Health Network Monotonicity (*)
For δ=1, all finite healthy networks converge to a Nash Network when following a round-robin best response.
Proof: Consider the function $f(g)=\sum_{j\in H} u_j(g)$. As all sensors in the healthy network, H, have a path to the sink, any better response adaptations (which are assured of preserving H by Theorem 8.5) are taken only if it decreases the sensor's cost. For δ=1 this adaptation causes no change in the utilities of the other sensors in H so $f(g)$ is a nondecreasing

function on a finite action space play must converge. Note that any profitable deviation increases the value of f.

We can then extend this result by incorporating the movement of sources into the healthy-network.

Theorem 8.8: Pruned Network Convergence(*)
For δ=1, all finite pruned sensor networks converge to a Nash Network under a round-robin best response.
Proof: By Theorem 8.6 after its first round-robin best response, pruned sensor networks consist of nodes either in H or disconnected. By Theorem 8.7, nodes in H converge to a Nash network. When profitable, nodes from not in H transition to H (and thus converge). Nodes not in H that can never profitably join H remain disconnected and thus are at their steady-state once disconnected.

Then combining this result with the earlier theorem, we can show that all finite sensor networks with a goal given by (8.6) and δ=1 converges to a Nash Network

Theorem 8.9: Arbitrary Network Convergence (*)
For δ=1 the sensor network scenario converges to a Nash network under a round-robin best response from every starting network.
Proof: By Theorem 8.6, every initial network necessarily converges under a round-robin best response to a pruned network. By Theorem 8.8 a finite pruned network necessarily converges under these conditions.

We can also characterize when a round-robin best response will result in a Nash network with no disconnected sensors.

Theorem 8.10: Guaranteed Paths to Sink (*)
Suppose it is possible to number the sensors such that $c\left(l_{k,k-1}\right) < b$ with the sink as sensor 0 and δ=1, then the round-robin best response network will converge to a network where all sensors have paths to the network.
Proof: Assume any initial distribution of links. After a single round of best responses, the network is a pruned network. As we assumed $c\left(l_{1,0}\right) < b$, the next best response of sensor 1 must place sensor 1 in the healthy network if it was not already. Likewise if node k is in H then node k+1 will join in its next iteration (if not before). By induction, all sensors in finite N must eventually join H implying that every sensor has a path to the sink.

Explicitly returning to the topic of this chapter, Theorem 8.9 supplies the sufficient condition to establish weak FIP.

Theorem 8.11: Sensor Networks and Weak FIP (*)
For $\delta=1$ the sensor network scenario has weak FIP.
Proof: Theorem 8.9 provides the requisite improvement path.

As Theorem 8.9 implies convergence from the empty network (which is a pruned network) and Theorem 8.11 implies that asynchronous timing can be employed with best response decision rules, a simple algorithm for autonomous sensor network formation can be written.

Algorithm 8.1: Sensor Network Formation

1) When first deployed, let all sensors (including the sink) in the network broadcast signals at the same power level.

2) Because it is at a common level, the initial signal should be sufficient for each sensor to calculate link gains to each of its detected sensors and thus the required transmit power to communicate and to estimate link formation costs. Let each sensor maintain the set N^{L_i} which is initialized to the set of sensors for which *i*'s cost of link formation is less than the benefit of a path to the sink.

3) At intervals determined by a random timer, each sensor *i* pings each sensors in its N^{L_i} to request its path to the sink, if one exists.

4) The pinging sensor either adds or switches a link to the lowest cost sensor that reports a path to the sink that does not pass through the pinging sensor and drops from N^{L_i} all sensors whose costs are greater than or equal to chosen link (thus there is no need to ping the sensor on the other end of the implemented link).

5) If $N^{L_i}=\varnothing$, the sensor is done as it has found its lowest cost path to the sink. Otherwise each sensor continues again from step 3) until the random timer has triggered a predetermined number of times in which case the sensor terminates this algorithm.

Because the system has weak FIP, rather than choosing the lowest cost link, each sensor could also be randomly choosing links and keeping those that improve its payoff. However, the best response algorithm should converge faster than the random better response algorithm.

8.4 Conclusions

This chapter introduced two different techniques for identifying when a game has weak FIP – showing that the game is a finite supermodular game and showing that there exists convergent improvement paths from all network states.

8.4.1 Analysis Summary

This chapter primarily considered the application of supermodular games to the analysis of cognitive radio networks. It was seen that a supermodular game always has at least one NE and that the NE of a game would form a lattice. However, identifying a supermodular game's NE requires that we solve the system of equations $a^* = \hat{B}(a^*)$ as we did in Chapter 4. By leveraging a relationship between standard interference functions and supermodular games, we established a novel condition on $\hat{B}(a)$ that ensures the uniqueness of an NE in a supermodular game and allows us to introduce a Lyapunov function for best response decision rules. Via adaptive dynamics, we also know that if the radios play best responses to observations over a finite history, a supermodular game will converge.

We then confirmed the claim in [Altman_03] that target SINR power control games are supermodular games by applying the concept of smooth supermodular games and extended this result to ad-hoc networks. As the power control game satisfied the condition to have a unique NE, the game has a Lyapunov function and is also stable. While a single reactive decision rule was simulated, the adaptive dynamics result of [Milgrom_90] informs us that the best responses also could have been based on previous observations as long as a finite history is employed.

We also studied sensor network formation and showed that for δ=1 the network has weak FIP. Unlike previous efforts, this relied on demonstrating that that from every action vector there exists an improvement path that converges to an NE. In general such an approach is not as straight forward as evaluating the second order conditions of supermodular games or potential games as evidenced by the seven theorems that had to be introduced to show convergence. However, game theory allows us to quickly identify other convergent decision rules based on the existence of weak FIP.

8.4.2 Design Implications

To an extent, supermodular games are closely related to the procedural analysis we performed in Chapter 3, particularly pseudo-contractions and standard interference functions. Partly this is because unlike potential games, not all myopic self-interested decision rules are guaranteed to converge under weak FIP or with a supermodular game and more specific decision rules have to be employed. Because of this limitation, cognitive radio networks which are supermodular games will generally not be appropriate for most ontological radio implementations. Instead, a cognitive radio should implement a specific decision rule such as its best response or a random better response (for finite supermodular games). In either case, this is best performed with a procedural radio though we see again that weak FIP implies that cognitive radios with properly designed random decision rules, e.g. a genetic algorithm cognitive radio, will be suitable.

As a properly best response algorithm will converge faster than a random better response, it is seen again that cognitive radios should incorporate the ability to perform scenario classification so that when a known scenario is encountered the best response algorithm can be employed and when an unknown scenario is encountered the cognitive radio uses a random better response as it converges under the broadest range of conditions.

As we saw in previous chapters, once we establish that a network satisfies a particular game model, it is trivial to develop low complexity convergent cognitive radio algorithms, as analysis of the game model in this document and elsewhere have yields lists of convergent algorithms. As was the case for both the power control and the sensor network formation examples, many of these algorithms are low complexity and well suited for use in rapidly deployed networks.

8.5 References

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Chapter 9: Conclusions

“<What good is it if it means nothing?>” - O. Card, Xenocide

In *Xenocide* four different sentient alien species come to inhabit the same planet, one of which is human at one point two aliens from different species discuss why humans dream. One completely fails to see the point while the other thinks that dreaming is the key to humanity’s creativity and thus to humanity’s success as a species.

This dissertation is intended to formalize the modeling and analysis of cognitive radio networks and from this formalization draw insights into how cognitive radios should be designed. Because of its heavy reliance on game theory, responses to this research over the last several years have almost exclusively fallen into two very different categories. Like the aliens contemplating dreaming in humans, either the research is viewed as absolutely critical to the success of cognitive radio or it is viewed as little better than a creative, but useless, intellectual exercise.

For many situations, the latter view has merit. What good is game theory if we can do the same analysis with traditional techniques? What algorithm has been designed with game theory that couldn’t be done using more traditional means? Why do we care about the goals of the radio at all? What does game theory bring to cognitive radio?

Strictly, every algorithm we analyzed in the previous four chapters could have been analyzed using more traditional techniques. We could’ve shown convergence of our algorithm in Chapter 7 using techniques from Zangwill’s convergence theorem. We could’ve shown the stability of the power control algorithms in Chapter 5 with only a direct application of Lyapunov’s theorem. We could show the existence of steady-states via more direct fixed point theorems. We could just as well cast each algorithm as a distributed controls (or dynamical systems) problem and use the evolution function and techniques of Chapter 3 to evaluate steady-states, convergence, and stability of the algorithm we are studying.

In response, we can point to the analysis and design benefits of a game theoretic listed in the following.

1) *Suitability for ontological and random procedural radios*

For ontological cognitive radios and procedural cognitive radios whose decision rules incorporate a degree of randomness, it is not generally possible to express the network behavior in terms of an evolution function needed for traditional analysis as the same input may produce very different outputs. Further, in the case of ontological radios we may only know the radio's available actions and its goal.

Lacking an evolution function, the dynamical systems and the contraction mappings approaches considered in Chapter 3 will be insufficient for modeling or analyzing these systems. At least for genetic algorithms, the network could be modeled and then analyzed using Markov models. However, any useful transition matrix would have to be determined empirically – the very process we are seeking to avoid. So if we were limited to traditional engineering analysis techniques, modeling and analyzing of most cognitive radio network behavior would be impractical.

Thus the techniques presented in this document permit us to analyze a problem which we could not handle with traditional approaches.

2) *More Efficient Analysis*

If we solve for a fixed point of an evolution function, we have identified a steady-state for a particular combination of decision rules. If a different combination of decision rules is deployed, then the analysis will need to be repeated.

On its own, this does not seem like a significant burden. But consider the deployment of cognitive radios in unlicensed bands – the location where cognitive radio (in the form of 802.11h) is already being deployed. One of the benefits of opening up unlicensed spectrum is that it permits the fielding of numerous different devices from different vendors which drive down prices for consumers. To differentiate their products different vendors typically employ different algorithms – permissible as wireless standards frequently do not specify radio resource management algorithms. For instance, while 802.22 specifies times within which a radio has to vacate a band when a primary user is detected, it does not currently specify the algorithm by which a new band is selected. Likewise 802.11h mandates DFS and TPC and specifies messages to support these operations, the algorithms by which frequency and power are adjusted are not specified.

Considering just 802.11h, in July 2006, the WiFi alliance already listed 13 different vendors and 72 different products for 802.11h. If the WiFi alliance, the FCC, or any radio designer wanted to ensure that all WiFi certified 802.11h products will not negatively interact, they would have some 2^{13} -1 combinations of decision rules to analyze if every vendor used their own algorithm.

Using game theoretic techniques, we only need to know the goals of the radios and the permissible actions, with the latter almost certainly defined as part of any standard, in order to determine the steady-states using the Nash equilibria concept. Thus instead of performing an 2^{13} -1 analyses for 802.11h, only a single analysis would need to be performed – meaning that only 1/8191 as much effort needs to be expended!

3) *Simplified Spectrum Management of Cognitive Radios*

The preceding implies another advantage of a game theoretic approach to analysis – simplified spectrum management. If the FCC or some primary spectrum holder specifies a particular combination of goals and allowable actions as part of a licensing agreement, then device testing can be simplified to merely verifying that the radio's algorithms act to increase one of the allowable goals.

Currently spectrum policy focuses solely on a specification of permissible actions (e.g., a spectral mask), but it seems likely that an additional specification of permissible goals would be sufficient to ensure acceptable performance while still permitting vendors and secondary users to use varying decision rules to differentiate their products.

Additionally, a predominately game theoretic approach permitted us to make the analysis and design insights listed in Sections 9.1 and 9.2. Section 9.3 reviews the research contributions presented in this dissertation, and the chapter concludes in Section 9.4 by identifying avenues for future research and describing additional planned publications.

9.1 Modeling and Analysis Summary

Using the model based approach to analysis proposed by this research, an analyst is able to immediately determine detailed information about steady-states, convergence, and stability simply by applying simple model identification criteria to the cognitive radio network. This will enable future cognitive radio network analysts to know within minutes the results this document showed over hundreds of pages. This approach should be able to cut months to years of man-hours off of the design cycle for novice cognitive radio algorithm designers (which virtually everyone is at this moment) and likely days for more experienced designers.

The set of models span all known cognitive radio implementation platforms (procedural, ontological, and nondeterministic procedural) and include dynamical systems, contraction mappings, standard interference functions, Markov models, potential games, and supermodular games. These models and the techniques for establishing if a cognitive radio network satisfies the conditions of the model are summarized in Table 9.1.

Table 9.1: Presented Models

Model	Basic model	Identification
Dynamical System	$\dot{a} = g(a,t)$, evolution equation $a(t^{k+1}) = d^t(a(t^k))$	$\dot{a} = g(a,t)$ always exists Solve g for d^t. d^t exists if g satisfies Picard-Lindelöf theorem
Contraction Mapping	$\|d(a),d(b)\| \le a\|a,b\|$ $\forall b,a \in A$	Blackwell's conditions
Standard Interference Function Power Control	$d_j(\mathbf{p}(t^k)) = p_j(t^k) I_j(\mathbf{p}(t^k))$	$I(\mathbf{p})$ satisfies positivity, montonicity, and scalability
Finite Ergodic Markov Chain	$P(a(t^{k+1}) = a^k \mid a(0),\ldots,a(t))$ $= P(a(t^{k+1}) = a^k \mid a(t^k))$	$\exists\, k$ such that $\mathbf{P}^k$ has all positive entries
Absorbing Markov Chain	$\mathbf{P}' = \left[\begin{array}{c\|c} \mathbf{Q} & \mathbf{R} \\ \hline \mathbf{0} & \mathbf{I}^{ab} \end{array}\right]$	Apply model definition
Normal Form Game	$\Gamma = \langle N, A\{u_j\}\rangle$	Map from cognition cycle
Mixed Strategy Strategic Form Game	$\Gamma = \langle N,\Delta(A),\{U_j\}\rangle$	Map from cognition cycle
Repeated Game	$\Gamma = \langle N, A\{u_j\},\{d_j\}\rangle$	Map from cognition cycle
Myopic Repeated Game	$\Gamma = \langle N, A\{u_j\},\{d_j\},T\rangle$	Map from cognition cycle
Potential Game	$\Delta u_i(a,b_i)$ everywhere related to $\Delta V(a,b_i)$	$\frac{\partial^2 u_i(a)}{\partial a_i \partial a_j} = \frac{\partial^2 u_j(a)}{\partial a_j \partial a_i}$ (others in Chapter)
Supermodular Game	(1) A_i is a complete lattice; (2) u_i is supermodular in a_i; and (3) u_i has increasing differences in (a_i, a_{-i})	(1) A_i is a closed interval in $\mathbb{R}^{k_i}$, (2) u_i is twice continuously differentiable on A_i (3) $\partial^2 u_i / \partial a_{ik} \partial a_{im} \ge 0$ for all $i \in N$ and all $1 \le k < m \le k_i$ (4) $\partial^2 u_i / \partial a_{ik} \partial a_{jm} \ge 0$ for all $i \ne j \in N, 1 \le k \le k_i$ and $1 \le m \le k_j$

For these game models, this chapter presented analysis insights that can be gleaned by demonstrating that a cognitive radio network satisfies the modeling conditions for one of the models listed in Table 9.1. The steady-state properties, the convergence properties, and the stability properties for each of these models are summarized in Table 9.2 Table 9.3, and Table 9.4, respectively. As we saw in Section 15.5.2, sometimes cognitive radio networks satisfy the conditions of multiple models. In these cases, the analytic insights from each of the applicable multiple models are available.

Table 9.2 Steady-State Properties by Model

Model	Existence	Identification
Dynamical System	Maybe, evaluate Leray-Schauder-Tychonoff theorem on evolution equation	Exhaustive Search Solve $a^* = d(a^*)$
Contraction Mappings	Yes (Banach's Theorem)	Recursion (Unique steady-state)
Standard Interference Function Power Control	Yes ([Yates_95])	Recursion (Unique steady-state), $\mathbf{Zp} = \vec{g}$
Finite Ergodic Markov Chain	Yes (Ergodocity theorem)	Recursion (Unique distribution), Solve $p^{*T}\mathbf{P} = p^{*T}$
Absorbing Markov Chain	Yes (Definition)	$p_{mm} = 1$
Normal Form Game	Maybe, evaluate Glicksberg-Fan theorem on cognitive radio goals	Exhaustive Search Solve $a^* = \hat{B}(a^*)$
Mixed Strategy Strategic Form Games	Yes for A finite (Nash's Fixed Point Theorem)	Solve $a^* = \hat{B}(a^*)$
Repeated Game	Maybe, evaluate Glicksberg-Fan theorem on cognitive radio goals, or evaluate for feasible enforced equilibria (numerous typically exist)	Exhaustive Search Solve $a^* = \hat{B}(a^*)$ Feasible enforced equilibrium
Myopic Repeated Game	Maybe, evaluate Glicksberg-Fan theorem on cognitive radio goals	Exhaustive Search Solve $a^* = \hat{B}(a^*)$
Potential Game	Yes, if A is compact and V bounded	$\arg\max_{a \in A} V(a)$
Supermodular Game	Yes	Exhaustive Search (must lie in a lattice) Solve $a^* = \hat{B}(a^*)$

Table 9.3: Convergence Properties by Model

Model	Sensitivity	Rate
Dynamical Systems	Apply Lyapunov's direct method (when possible)	No general technique
Contraction Mappings	Everywhere convergent	$\left\| a\left(t^k\right), a^* \right\| \le \frac{a^k}{1-a} \left\| a\left(t^1\right), a\left(t^0\right) \right\|$
Standard Interference Function Power Control	Everywhere convergent	$\left\| \mathbf{p}\left(t^k\right), \mathbf{p}^* \right\| \le a^k \left\| \mathbf{p}(0), \mathbf{p}^* \right\|$
Finite Ergodic Markov Chain	Converges to distribution from all starting distributions	Transition matrix dependent
Absorbing Markov Chain	$\mathbf{B} = \mathbf{NR}$	$\mathbf{t} = \mathbf{N}\,\mathbf{1}$
Normal Form Game	Convergence not defined	Convergence not defined
Mixed Strategy Strategic Form Games	Convergence not defined	Convergence not defined
Repeated Game	Assumes no adaptations	Assumes no adaptations
Myopic Repeated Game	Apply IESDS, FIP, weak FIP	Length of longest improvement path
Potential Game	All autonomously rational decision rules converge	Length of longest improvement path
Supermodular Game	All locally optimal decision rules converge	Length of longest improvement path

Table 9.4: Stability Properties by Model

Model	Lyapunov Stability	Attractivity
Dynamical Systems	Apply Lyapunov's direct method (when possible)	Apply Lyapunov's direct method (when possible)
Contraction Mappings	Global	Global
Standard Interference Function Power Control	Global	Global
Finite Ergodic Markov Chain	No	No
Absorbing Markov Chain	No	Only if unique steady-state
Normal Form Game	Stability not defined	Stability not defined
Mixed Strategy Strategic Form Games	Stability not defined	Stability not defined
Repeated Game (assuming correct differentiation of punishment and deviation)	Yes	Yes
Myopic Repeated Game	Not implicit to model	Not implicit to model
Potential Game	Isolated potential maximizers are Lyapunov stable for all rational decision rules.	Attractive to potential maximizers if finite action space or finite step size.
Supermodular Game	Best response decision rules if unique NE	Best response decision rules if unique NE

We also presented two different model independent approaches to determining the desirability of network behavior – Pareto optimality and evaluation of a network objective function. Showing that a network state is Pareto optimal was shown to be of less value than demonstrating that the state maximized the intended network objective function.

We must also acknowledge certain analytical difficulties that arise when information is limited. We may not be able to precisely describe a network's evolution function pre-deployment if decision processes and goals evolve to better reflect a user's preferences. A radio's available actions may also evolve in time to incorporate new waveforms that we could not anticipate ahead of time. From an analysis perspective, this situation can be analogized to attempting to solve a system of equations of unknown order with unknown coefficients and an unknown number of variables. This indicates that much caution should be taken before deploying radios for which we do not know a priori the radios' actions or the goals. Perhaps, it will be possible to broadly classify the decision update processes action sets, and goals based on what is known about the implementation of the radios. In which case a game theoretic preference approach should be able to address this situation, but barring this condition, analysis of such a system currently appears intractable.

9.2 Design Summary

Leveraging the modeling and analysis techniques, we were able to develop new algorithms, design guidelines, and insights into cognitive radio design issues. Specific algorithms for general waveform adaptation, power control, and sensor network formation were proposed, analyzed, and shown to have desirable steady-state, convergence and stability properties.

The proposed interference reducing network (IRN) design framework ensures that loner radios (procedural or ontological) converge to an interference minimizing steady state under the scenarios of global altruism, local altruism, isolated clusters, close proximity, and with controlled observations. This framework achieves these results by ensuring that the network constitutes an exact potential game whose potential function is a negated scalar multiple of the sum network interference, thereby allowing us to leverage the steady-state, convergence, and stability results of Chapter 5 and adding an assurance of a desirable equilibrium.

The global and local altruism IRNs wherein each radio's interference minimization goal incorporates other radios' interference observations were seen to be applicable to any waveform adaptation algorithm, but to scale badly. By ensuring that the network satisfied bilateral symmetric interference, it was seen that loner radios would implement an IRN for the isolated cluster, close proximity, and controlled observation scenarios. These last three scenarios require no coordination between cognitive radios to ensure interference minimization implying that in these scenarios low network overhead, low device complexity algorithms realize an IRN.

It was seen that waveform adaptation algorithms will frequently have non-isolated NE, so stabilization requires the radios employ ε-better response algorithms. It was also seen that if certain assumptions were made about legacy systems then the network consisting of legacy radios and cognitive radios would still comprise an interference reducing network and that even when these assumptions fail, the self-interested adaptations of the cognitive radios would generally reduce the interference experienced by the legacy systems.

The IRN design framework was leveraged to develop a new dynamic frequency selection (DFS) algorithm for 802.11h. By requiring the access points to observe the RTS/CTS messages of other access points to guide their decision process, bilateral symmetric interference is achieved and an IRN results without any additional coordination between access points and as long as the access points act to reduce their own observed interference. It was analytically shown that this algorithm performs well under a variety of relaxed assumptions including policy variations, the presence of legacy radios, noise corrupted observations, private frequency preferences, and asynchronous timings, and empirically shown to perform well different access points transmit their RTS/CTS signals at different power levels within the same channel.

Numerous design inferences were also drawn from analysis. It was seen that a network of myopic loner radios cannot be guaranteed to converge under autonomous rationality unless the underlying game has weak FIP. It was seen that guaranteeing the convergence of arbitrary ontological radios under autonomous rationality requires the underlying game to have FIP. Potential games ensure permit the lowest complexity loner radio algorithms to converge. For radios with a finite set of available adaptations, it was seen that incorporating randomness into the decision rules ensures convergence over the broadest set of conditions but that well designed procedures will generally converge faster. This implies that the genetic algorithm implementation approaches being considered by various researchers will be an excellent choice when the radio must operate in broad set of scenarios.

The ubiquitous presence of unbounded noise informs us that cognitive radio networks will always constitute ergodic Markov chains, but we do know identified NE will still have a relatively higher probability of being occupied and that frequently many states will have an extremely small probability of being occupied making the occupancy of certain states a theoretical, though generally not a practical, concern. Noise, however, is a more serious problem for social radios which try to influence the adaptations of other radios via punishment as noise ensures eventual catastrophic failure if the network does not include some additional mechanism to differentiate between punishment and deviation. It was also seen that the ability to negotiate will be critical to the deployment of social radios due to the variances of goals and adaptations likely to be encountered.

9.3 Research Contributions

The primary goal of this research was to develop a methodology for analyzing the interactions of cognitive radios with a particular interest in addressing the identification of steady-states, the optimality of those steady-states, the conditions for convergence, and the stability of the cognitive radio algorithms. Achieving this goal required the refinement of game theoretic concepts and techniques, the identification of typical cognitive radio applications that satisfy the conditions of these models, and the development of simulations to verify the analytic results implied by this methodology.

Original research contributions are made in every chapter in this dissertation and are highlighted in Table 9.5.

Table 9.5: Major Novel Contributions Made as Part of this Work

Chapter	Research Contributions
Chapter 1	Definition of procedural and ontological cognitive radios. Definition of waveform
Chapter 2	General model of cognitive radio interactions
Chapter 3	Application of dynamical systems to the analysis of procedural radios Stability of standard interference function (SIF) Application of SIF to ad-hoc networks
Chapter 4	Application of game theory to cognitive radios General game model of cognitive radio networks Novel random better response algorithm with broader convergence conditions Convergence analysis for basic game theoretic properties under different

	decision timings Ergodic Markov chain model of noisy cognitive radio networks Necessary condition for convergence of myopic rational cognitive radios
Chapter 5	Application of potential games to wireless network design Multilateral Symmetric Interference Games Identification of ordinal potential games via better response transformations Convergence of round-robin/random better response algorithms for potential games with infinite action spaces Convergence of asynchronous better response algorithms for finite action spaces Stability of potential games for discrete time adaptations
Chapter 6	Interference Reducing Network (IRN) design framework Global altruism algorithm Local altruism algorithm Bilateral Symmetric Interference identification condition General algorithm for implementing an IRN in an isolated cluster Close proximity algorithm Impact of legacy devices
Chapter 7	Novel Dynamic Frequency Selection algorithm for ad-hoc networks and its performance under non-ideal circumstances
Chapter 8	Condition for uniqueness and stability of supermodular games A convergence proof of typical ad-hoc TPC algorithms Novel sensor network formation algorithm

Another objective of this research was developing, standardizing, and popularizing techniques for analyzing and designing cognitive radio networks. Beyond influencing the direction of many other researchers at Virginia Tech, this work has also had a significant impact on the work of cognitive radio researchers throughout the world. The following sections list the publications generated as part of this work and external citations of these publications.

9.3.1 Publications

The following is a listing of publications that have been created as part of the research presented in this document. The list includes two award winning papers, one chapter in the first textbook on cognitive radio, two journal papers, and one magazine article.

1) [submitted] J. Neel, R. Menon, A. MacKenzie, J. Reed, R. Gilles, "Interference Reducing Networks," submitted to *IEEE JSAC on Adaptive, Spectrum Agile, and Cognitive Wireless Networks.*

2) [accepted] J. Neel, J. Reed, "Situational Awareness for Cognitive Radios," Submitted to *SDR Forum 2006*.

3) [accepted] J. Neel, J. Reed, Performance of Distributed Dynamic Frequency Selection Schemes for Interference Reducing Networks," Accepted to *Milcom 2006* Oct. 23-25, 2006.

4) J. Neel, J. Reed, A. MacKenzie, "Analyzing Cognitive Radio Networks" in **Cognitive Radio Technology**, ed. B. Fette, Elsevier Publications, August 11, 2006.

5) V. Srivastava, J. Neel, A. MacKenzie, J. Hicks, L.A. DaSilva, J.H. Reed and R. Gilles, "Using Game Theory to Analyze Wireless Ad Hoc Networks," *IEEE Communications Surveys and Tutorials* 4th quarter 2005, vol. 7, no 4, pp. 46-54.

6) J. Neel, "Game theory can be used to analyze cognitive radio," *EE Times*, August 29, 2005.

7) J. Neel, R. Menon, A. MacKenzie, J. Reed, "Using Game Theory to Aid the Design of Physical Layer Cognitive Radio Algorithms," accepted on basis of abstract to *Conference on Economics, Technology and Policy of Unlicensed Spectrum*, May 16-17 2005, Lansing, Michigan.

8) J. Hicks, A. MacKenzie, J. Neel, J. Reed, "A Game Theory Perspective on Interference Avoidance," *Globecom 2004*, November 29 - December 3, 2004.

9) [Named outstanding paper] J. Neel, J. Reed, R. Gilles, "Game Models for Cognitive Radio Analysis," *SDR Forum 2004 Technical Conference*, November 2004.

10) J. Neel, J. Reed, and R. Gilles, "Convergence of Cognitive Radio Networks," *WCNC2004*, March 25, 2004.

11) S. Ginde, R. Buehrer, and J. Neel, "A Game Theoretic Analysis of the GPRS Adaptive Modulation Schemes," *Fall VTC 2003*.

12) [Named outstanding paper] J. Neel, J. Reed, R. Gilles, "The Role of Game Theory in the Analysis of Software Radio Networks," *SDR Forum Technical Conference* November, 2002.

13) J. Neel, R. Buehrer, J. Reed, and R. Gilles, "Game Theoretic Analysis of a Network of Cognitive Radios," *Midwest Symposium on Circuits and Systems 2002*.

9.3.2 External Citations

Despite having a very narrow window for citations to appear, the publications generated as part of this research have already been cited in several publications, classes, and proposals. The following lists works generated as part of this project and publications that cited those works from authors external to Virginia Tech – a reasonable metric for determining the extent which this research is influencing others' research around the world.

Publication: J. Neel, R. Menon, A. MacKenzie, J. Reed, "Using Game Theory to Aid the Design of Physical Layer Cognitive Radio Algorithms," accepted on basis of abstract to *Conference on Economics, Technology and Policy of Unlicensed Spectrum*, May 16-17 2005, Lansing, Michigan.

1) W. Lehr, "Managing shared access to a spectrum commons," *DySPAN 2005*, pp. 420-444, Nov. 8-11, 2005.

Publication: J. Hicks, A. MacKenzie, J.Neel, J. Reed, "A game theory perspective on interference avoidance", *GLOBECOM 2004*, pp. 257-261.

2) A. Fridman, R. Grote, S. Weber, K. Dandekar, M. Kam, "Robust optimal power control for ad hoc networks," *2006 Conference on Information Sciences and Systems*, Princeton University, March 22–24, 2006.
3) C. Liang, K. Dandekar, "Power Management in MIMO Ad Hoc Network: A Game-Theoretic Approach," Submitted to *IEEE Transactions on Wireless Communications* (http://www.ece.drexel.edu/faculty/dandekar/Papers/Liang_WirelessComm05.pdf)

Publication: J. Neel, J. H. Reed, R. P. Gilles, "Game Models for Cognitive Radio Algorithm Analysis," *SDR Forum 2004 Technical Conference*, November, 2004.

4) R. Nuti, "Criteri distribuiti di allocazione delle risorse nelle reti wireless ad hoc," PhD Dissertation, University of Pisa, Oct 2005.

Publication: J. Neel, J. Reed, R. Gilles "Convergence of Cognitive Radio Networks," *Wireless Communications and Networking Conference*, March 2004.

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9.4 Future Work

While this document made an extensive presentation of techniques for the modeling, analysis, and design of cognitive radio networks, this is far from an exhaustive treatment of all cognitive radio algorithms. This section presents topics of research that still should be addressed beyond the interference avoidance, node participation, and topology control applications being developed for the ONR project and the ongoing cognitive radio research a Virginia Tech and additional publications planned based on the material presented in this document.

9.4.1 Research Topics

The following lists some of the research topics identified in the body of this document that merit further research.

Joint power/frequency adaptation

The target SINR utility function appears to be an attractive algorithm for power control and frequency adaptations. To date all simulations of these algorithms have converged under numerous decision rules, but no theoretical basis for this is known. It may be that as we saw with DFS and legacy radios that there exist conditions that confound application of the game models but that these conditions are empirically rare.

Modeling and analyzing network routing

To an extent the queues used in the routers in the internet can be analogized to facilities which under IPV4 all devices experience approximately equally. Loosely, the choice of routes dictates the choice of "facilities" indicating that this problem should be susceptible to potential game analysis.

Asymmetric potential games

Designing the IRN framework and the IRN DFS algorithms required that we identify and exploit symmetry conditions implicit to BSI games. To make such an approach more generalizable, it would be valuable to characterize how much asymmetry can be introduced to BSI and congestion games while still preserving the important convergence and stability properties of potential games. For example under what conditions would positive correlation of interaction terms or facility benefits be sufficient to imply an ordinal potential game? Would scaling these terms by a common factor yield a weighted potential game?

Cross-layer algorithms

In general the work presented in this document (and the work being performed as part of the ONR project) focused on single layer and single parameter algorithms. In general, we can treat adaptations over numerous parameters simply as more complex actions and then apply the techniques presented in this document. However, this research will likely be delayed by the immaturity of theoretical expressions of cross-layer problems as without a well-defined mapping from actions to outcomes, analysis is not possible.

Standards Applications

Development of algorithms that explicitly account for the implementation details of specific standards should hasten the deployment of cognitive radios. For instance, this could consider DFS / power control in an 802.22 setting and network formation algorithms for 802.11s.

Scenario Classification

As we saw different operational scenarios can be modeled as different game models which imply different algorithms are preferable. The capacity to recognize which scenario a radio is operating under should significantly enhance the performance of cognitive radio networks.

Bargaining Algorithms for Cognitive Radio

As we saw in Chapter 4, the implementation of punishment algorithms in cognitive radio networks should be accompanied by a negotiation capability because of the diverse combination of goals, operating conditions, and capabilities. Properly designed bargaining algorithms could simplify this problem and enhance the attractiveness of social radio networks.

Differentiating punishment from deviation

Punishment algorithms are doomed to catastrophic failure when they are unable to differentiate between punishments and deviations from an agreed operating point. Development of generalized and specific techniques for performing this will also enhance the attractiveness of social radio networks.

9.4.2 Planned Publications

The following is a listing and brief description of publications intended for submission to either a journal or a magazine after submission of this document based on material presented in this document.

1) *"Novel Dynamic Frequency Selection Algorithms"*

   This will be an extended treatment of the Milcom DFS paper including the additional material presented in Chapter 7.

2) *"Game Theoretic Insights into the Design of Cognitive Radio Networks"*

   Throughout this document, game theory has been leveraged to provide valuable insights into the design of cognitive radio networks. These will be collected into a single document with justification of these insights.

3) *"Potential Games in Wireless Networks"*

This paper will summarize the various insights that can be gained by applying potential games to cognitive radio networks and present some of the applications included in this report such as DFS, power control and waveform adaptation as well as others not included in this document such as network selection.

4) *"An Algorithm for Distributed Sensor Network Formation"*

This paper will present the analytic sensor network formation presented in Chapter 8.

5) *"Convergence and Stability of Games with FIP and Weak FIP"*

Intended for an economics journal, this research necessitated the development of numerous new results related to the convergence and stability implications of FIP, weak FIP, potential games, and supermodular games.

6) *"Identification of Ordinal Potential Games"*

Intended as an economics letter, this will present the novel technique for identifying ordinal potential games developed as part of this research presented in Chapter 5.

Additionally, this document is intended to serve as the core for a text book on modeling, analyzing, and designing cognitive radio networks.

Vitae

James "Jody" O'Daniell Neel received his B.S.E.E. and M.S.E.E. degrees from Virginia Tech in 1999 and 2001, respectively and is a PhD Candidate and IREAN Fellow at Virginia Polytechnic Institute and State University's (Virginia Tech) Mobile and Portable Radio Research Group (MPRG). He has an extensive history of involvement with software radio, with twenty publications on software radio, including two textbook chapters on data conversion and the history of software radio in Dr Reed's **Software Radio: A Modern Approach to Engineering** and another chapter on analyzing cognitive radio interactions in the textbook **Cognitive Radio Technology** edited by Dr Bruce Fette. Jody also received awards for his 2002 and 2004 SDR Forum papers on applying game theory to the analysis of cognitive radio and was a session chair for the 2004 and 2005 SDR Forum Technical conference sessions on cognitive radio and regularly serves as an SDR industry consultant.

In 2002, Jody received his Master of Science degree in Electrical Engineering from Virginia Tech for work demonstrating the feasibility of implementing a software radio on a Custom Computing Machine (CCM) processor platform. This work included the development of tools for programming and emulating the CCM, the implementation of waveforms on the CCM, and the design and implementation of a FPGA-based controller for managing the CCM. During his Masters, Jody also developed the course material for DSP Implementation of Communications Systems.

Jody's research interests include cognitive radio, software radio, game theory, signal processing theory, communications system design and implementation, processor architectures and tradeoffs, and commercial standards.